Miniversal deformations of pairs of skew-symmetric matrices under congruence

by

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Abstract

Miniversal deformations for pairs of skew-symmetric matrices under congruence are constructed. To be precise, for each such a pair (A, B) we provide a normal form with a minimal number of independent parameters to which all pairs of skew-symmetric matrices $(\widetilde{A}, \widetilde{B})$, close to (A, B) can be reduced by congruence transformation which smoothly depends on the entries of the matrices in the pair $(\widetilde{A}, \widetilde{B})$. An upper bound on the distance from such a miniversal deformation to (A, B) is derived too. We also present an example of using miniversal deformations for analyzing changes in the canonical structure information (i.e. eigenvalues and minimal indices) of skew-symmetric matrix pairs under perturbations.

Keywords: Skew-symmetric matrix pair, Skew-symmetric matrix pencil, Congruence canonical form, Congruence, Perturbation, Versal deformation
2000 MSC: 15A21, 15A63

1. Introduction

Canonical forms of matrices and matrix pencils, e.g., Jordan and Kronekher canonical forms, are well known and studied with various purposes but the reductions to these forms are unstable operations: both the corresponding canonical forms and the reduction transformations depend discontinuously on the entries of an original matrix or matrix pencil. Therefore, V.I. Arnold introduced a normal form, with the minimal number of independent parameters, to which an arbitrary family of matrices $\tilde{A}$ close to a given matrix A can be reduced by similarity transformations smoothly depending on the entries of $\tilde{A}$. He called such a normal form a miniversal deformation of A. Now the notion of miniversal deformations has been extended to matrices with respect to congruence [14] and *congruence [15], matrix pencils with respect to strict equivalence [19, 23] and congruence [11], etc. (more detailed list is given in the introduction of [15]).

Miniversal deformations can help us to construct stratifications, i.e., closure hierarchies, [13, 20, 21] of orbits and bundles. These stratifications are the graphs that show which canonical forms the matrices (or matrix pencils) may have in an arbitrarily small neighbourhood of a given matrix (or matrix pencil). In particular, the stratifications show how the eigenvalues may coalesce or split apart, appear or disappear. Both the stratifications and miniversal deformations may be useful when the matrices arise as a result of measures and their entries are known with errors, see [27, 30] for some applications in control and stability theory.

The questions related to eigenvalues and another canonical information for the pencils $A - sB$, where $A = \pm A^T$ and $B = \pm B^T$, or $A = \pm A^*$ and $B = \pm B^*$, dragged some attention over time and, especially, recently, e.g., see the following papers on canonical forms [35, 37], codimension computations [8, 9, 17, 18], low rank perturbations [4], miniversal deformations [11, 14, 23], partial [13, 22] and general [16] stratification results, staircase forms [5, 7]. Such pencils also appear as the structure preserving linearizations of the corresponding matrix polynomials [32, 33]. In particular, the papers [4, 16, 17, 35, 37] deal with skew-symmetric matrix pencils, i.e. $A - sB$, where $A = -A^T$ and $B = -B^T$, and [33] deals with skew-symmetric matrix polynomials. Skew-symmetric matrix pencils appear in multisymplectic partial differential equations [6], systems with bi-Hamiltonian structure [34], as well as in the design of a passive velocity field controller [28]. Recall that, an $n \times n$ skew-symmetric matrix pencil $A - sB$ is called congruent to $C - sD$ if and only if there is a non-singular matrix S such that $S^T A S = C$ and $S^T B S = D$. The set of matrix pencils congruent to a skew-symmetric matrix pencil $A - sB$ is called a congruence orbit of $A - sB$.

In this paper, we derive the miniversal deformations of skew-symmetric matrix pencils under congruence and bound the distance from these deformations to unperturbed matrix pencils in terms of the norm of the perturbations. The number of independent parameters in the miniversal deformations is equal to the codimensions of the congruence orbits of skew-symmetric matrix pencils (obtained independently in [17]). The Matlab functions for computing these codimensions were developed [12] and added to the Matrix Canonical Structure (MCS) Toolbox [25]. Example 2.1 shows how the miniversal deformations from Theorem 2.1 can be used for the investigation of the possible changes of the canonical structure information.

The rest of the paper is organized as follows. In Section 2, we present the main theorems, i.e., we construct miniversal deformations of skew-symmetric matrix pencils and prove an upper bound on the distance between a skew-symmetric matrix pencil and its miniversal deformation. In Section 3, we describe the method of constructing deformations (Section 3.1) and derive

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the miniversal deformations step by step: for the diagonal blocks (Section 3.2), for the off-diagonal blocks that correspond to the canonical summands of the same type (Section 3.3), and for the off-diagonal blocks that correspond to the canonical summands of different types (Section 3.4).

In this paper all matrices are considered over the field of complex numbers. Except in Example 2.1, we use the matrix pair notation (A, B) instead of the pencil notation $A - sB$. We also use one calligraphic letter, e.g., $\mathcal{A}$ or $\mathcal{D}$, to refer to a matrix pair.

2. The main results

In this section, we present the miniversal deformations of pairs of skew-symmetric matrices under congruence and obtain an upper bound on the distance between a skew-symmetric matrix pair and its miniversal deformations. In Section 3, we will derive these miniversal deformations.

First we recall the canonical form of pairs of skew-symmetric matrices under congruence given in [37]. For each $k = 1, 2, \ldots,$ define the $k \times k$ matrices

$$
J_k(\lambda) := \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}, \qquad I_k := \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix},
$$

where $\lambda \in \mathbb{C}$, and for each $k = 0, 1, \ldots,$ the $k \times (k+1)$ matrices

$$
F_k := \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \qquad G_k := \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}.
$$

All non-specified entries of the matrices $J_k(\lambda), I_k, F_k,$ and G_k are zero.

Lemma 2.1 ([36, 37])**.** *Every pair of skew-symmetric complex matrices is congruent to a direct sum, determined uniquely up to permutation of summands, of pairs of the form*

$$
\mathcal{H}_n(\lambda) := \left(\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} \right), \quad \lambda \in \mathbb{C}, \tag{1}
$$

$$
\mathcal{K}_n := \left(\begin{bmatrix} 0 & J_n(0) \\ -J_n(0)^T & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \right), \tag{2}
$$

$$
\mathcal{L}_n := \left(\begin{bmatrix} 0 & F_n \\ -F_n^T & 0 \end{bmatrix}, \begin{bmatrix} 0 & G_n \\ -G_n^T & 0 \end{bmatrix} \right). \tag{3}
$$

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Thus, each pair of skew-symmetric matrices is congruent to a direct sum

$$(A, B)_{\mathrm{can}} = \bigoplus_{i=1}^{a} \mathcal{H}_{h_i}(\lambda_i) \oplus \bigoplus_{j=1}^{b} \mathcal{K}_{k_j} \oplus \bigoplus_{r=1}^{c} \mathcal{L}_{l_r}, \tag{4}$$

consisting of direct summands of three types of pairs.

2.1. Miniversal deformations

The concept of a miniversal deformation of a matrix with respect to similarity was given by V. I. Arnold [1] (see also [3, § 30B]). This concept can straightforwardly be extended to pairs of skew-symmetric matrices with respect to congruence.

A *deformation* of a pair of skew-symmetric $\hat{n} \times \hat{n}$ matrices (A, B) is a holomorphic mapping $\mathcal{A}(\vec{\delta})$, where $\vec{\delta} = (\delta_1, \ldots, \delta_k)$, from a neighborhood $\Omega \subset \mathbb{C}^k$ of $\vec{0} = (0, \ldots, 0)$ to the space of pairs of skew-symmetric $\hat{n} \times \hat{n}$ matrices such that $\mathcal{A}(\vec{0}) = (A, B)$. Note that in this paper we consider only skew-symmetric deformations, i.e., the skew-symmetric structure of matrix pairs is preserved. Therefore we write only "deformation" but not "skew-symmetric deformation" without the risk of confusion.

Definition 2.1. A deformation $\mathcal{A}(\delta_1, \ldots, \delta_k)$ of a pair of skew-symmetric matrices (A, B) is called *versal* if for every deformation $\mathcal{B}(\sigma_1, \ldots, \sigma_l)$ of (A, B) we have

$$\mathcal{B}(\sigma_1, \ldots, \sigma_l) = I(\sigma_1, \ldots, \sigma_l)^T \mathcal{A}(\varphi_1(\vec{\sigma}), \ldots, \varphi_k(\vec{\sigma})) I(\sigma_1, \ldots, \sigma_l),$$

where $I(\sigma_1, \ldots, \sigma_l)$ is a deformation of the identity matrix, and all $\varphi_i(\vec{\sigma})$ are convergent in a neighborhood of $\vec{0}$ power series such that $\varphi_i(\vec{0}) = 0$. A versal deformation $\mathcal{A}(\delta_1, \ldots, \delta_k)$ of (A, B) is called *miniversal* if there is no versal deformation having less than k parameters.

By a $(0, *)$ *matrix* we mean a matrix whose entries are 0 and $*$ and we consider pairs $\mathcal{D}$ of $(0, *)$ matrices. We say that a pair of skew-symmetric matrices *is of the form* $\mathcal{D}$ if it can be obtained from $\mathcal{D}$ by replacing the stars with complex numbers, respecting the skew-symmetry. Denote by $\mathcal{D}(\mathbb{C})$ the space of all pairs of skew-symmetric matrices of the form $\mathcal{D}$, and by $\mathcal{D}(\vec{\varepsilon})$ the pair of parametric skew-symmetric matrices obtained from $\mathcal{D}$ by replacing the (i, j)-th and (j, i)-th stars with the parameters ε_{ij} and $-\varepsilon_{ji}$, respectively, in the first matrix and the (i', j')-th and (j', i')-th stars with the parameters $\varepsilon'_{i'j'}$ and $-\varepsilon'_{j'i'}$, respectively, in the second matrix. In other words

$$\mathcal{D}(\vec{\varepsilon}) := \Big(\sum_{(i,j)\in\mathrm{Ind}_1(\mathcal{D})} \varepsilon_{ij} E_{ij}, \sum_{(i',j')\in\mathrm{Ind}_2(\mathcal{D})} \varepsilon'_{i'j'} E_{i'j'} \Big), \tag{5}$$

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$$\mathcal{D}(\mathbb{C}) := \left\{ \mathcal{D}(\vec{\varepsilon}) \mid \vec{\varepsilon} \in \mathbb{C}^k \right\} = \left\{ \left(\underset{(i,j)\in\mathrm{Ind}_1(\mathcal{D})}{+} \mathbb{C}E_{ij}, \underset{(i',j')\in\mathrm{Ind}_2(\mathcal{D})}{+} \mathbb{C}E_{i'j'} \right) \right\}, \qquad (6)$$

where

$$\mathrm{Ind}_1(\mathcal{D}), \mathrm{Ind}_2(\mathcal{D}) \subseteq \{1,\dots,\hat{n}\} \times \{1,\dots,\hat{n}\},$$

are the sets of indices of the stars in the upper-triangular parts of the first and the second matrices, respectively, of the pair $\mathcal{D}$, and E_{ij} is the matrix whose (i,j)-th entry is 1, (j,i)-th entry is -1 and the other entries are zero. Note that the large "$+$" in (6) denotes the entrywise sum of matrices.

Following [23], we say that a miniversal deformation of (A,B) is *simplest* if it has the form $(A,B) + \mathcal{D}(\vec{\varepsilon})$, where $\mathcal{D}$ is a pair of $(0, *)$ matrices. If the matrix pair $\mathcal{D}$ in $(A,B) + \mathcal{D}(\vec{\varepsilon})$ has no zero entries (except on the main diagonals), then $\mathcal{D}$ defines the deformation

$$\mathcal{U}(\vec{\varepsilon}) := \left(A + \sum_{i=1}^{\hat{n}} \sum_{j=i+1}^{\hat{n}} \varepsilon_{ij} E_{ij}, \ B + \sum_{i=1}^{\hat{n}} \sum_{j=i+1}^{\hat{n}} \varepsilon'_{ij} E_{ij} \right). \qquad (7)$$

In other words, for all pairs of $\hat{n} \times \hat{n}$ skew-symmetric matrices $(A+E, B+E')$ that are close to a given pair of skew-symmetric matrices (A,B), we derive the normal form $\mathcal{A}(E,E')$ with respect to the congruence transformation

$$(A+E, B+E') \mapsto S(E,E')^T (A+E, B+E') S(E,E') =: \mathcal{A}(E,E'), \qquad (8)$$

in which $S(E,E')$ is holomorphic at 0 (i.e. its entries are power series in the entries of E and E' that are convergent in a neighborhood of 0) and $S(0,0)$ is a nonsingular $\hat{n} \times \hat{n}$ matrix.

Since $\mathcal{A}(0,0) = S(0,0)^T(A,B)S(0,0)$, we can take $\mathcal{A}(0,0)$ equal to the congruence canonical form $(A,B)_{\mathrm{can}}$ of (A,B), see (4). Then

$$\mathcal{A}(E,E') = (A,B)_{\mathrm{can}} + \mathcal{D}(E,E'), \qquad (9)$$

where $\mathcal{D}(E,E')$ ($= \mathcal{D}(\vec{\varepsilon})$ for some $\vec{\varepsilon} \in \mathbb{C}^k$) is a pair of skew-symmetric matrices that is holomorphic at 0 and $\mathcal{D}(0,0) = (0,0)$. In Theorem 2.1 we derive $\mathcal{D}(E,E')$ with the minimal number of nonzero entries that can be attained using the congruence transformation defined in (8).

We use the following notation, in which each star denotes a function of the entries of E and E' that is holomorphic at zero:

- 0_{mn} is the $m \times n$ zero matrix;

- 0_{mn*} is the $m \times n$ matrix $\begin{bmatrix} & & 0 \\ 0_{m-1,n-1} & & \vdots \\ & & 0 \\ 0 & \dots & 0 & * \end{bmatrix}$;

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- $0_{mn}^{\nwarrow}$ is the $m \times n$ matrix

$$\begin{bmatrix} * & \\ * & \\ \vdots & 0_{m,n-1} \\ * & \end{bmatrix} \quad \text{if } m \leqslant n, \text{ and} \quad \begin{bmatrix} * & * & \dots & * \\ & 0_{m-1,n} & \end{bmatrix} \quad \text{if } m \geqslant n \qquad (10)$$

(if $m = n$, then we can take any of the matrices defined in (10));

- $0^{\nearrow}$, $0^{\searrow}$ and $0^{\swarrow}$ are matrices that are obtained from $0^{\nwarrow}$, by clockwise rotation by 90°, 180° and 270°, respectively;

- $0_{mn}^{\leftarrow}$ is the $m \times n$ matrix $\begin{bmatrix} * & \\ \vdots & 0_{m,n-1} \\ * & \end{bmatrix}$

(in contrast to $0_{mn}^{\nwarrow}$ and $0_{mn}^{\swarrow}$, the matrix $0_{mn}^{\leftarrow}$ has stars in the first column even if $m > n$);

- $0_{mn}^{\rightarrow}$ is the $m \times n$ matrix $\begin{bmatrix} & & * \\ 0_{m,n-1} & & \vdots \\ & & * \end{bmatrix}$

(in contrast to $0_{mn}^{\nearrow}$ and $0_{mn}^{\searrow}$, the matrix $0_{mn}^{\rightarrow}$ has stars in the last column even if $m > n$);

- $0_{mn}^{\urcorner}$ is the $m \times n$ matrix $\begin{bmatrix} * & & \\ \vdots & 0_{m-1,n-1} & \\ * & \dots & & * \end{bmatrix}$;

- $0_{mn}^{\boxminus}$ with $m < n$ is the $m \times n$ matrix

$$\begin{bmatrix} 0 & \dots & 0 & & 0 \\ \vdots & & \vdots & & \\ 0 & \dots & 0 & * \dots * & 0 \end{bmatrix} \qquad (n - m \text{ stars})$$

if $m \geq n$ then $0_{mn}^{\boxminus} = 0$.

Further, we will usually omit the indices m and n.

Let

$$(A, B)_{\text{can}} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_t \qquad (11)$$

be a canonical pair of skew-symmetric complex matrices for congruence, in which $\mathcal{X}_1, \dots, \mathcal{X}_t$ are pairs of the form (1)–(3), and let $\mathcal{D}(E, E')$ be a pair of skew-symmetric matrices, defined in (9), whose matrices are partitioned into blocks conformally to the decomposition (11):

$$\mathcal{D}(E, E') = \mathcal{D} = \left(\begin{bmatrix} D_{11} & \dots & D_{1t} \\ \vdots & \ddots & \vdots \\ D_{t1} & \dots & D_{tt} \end{bmatrix}, \begin{bmatrix} D'_{11} & \dots & D'_{1t} \\ \vdots & \ddots & \vdots \\ D'_{t1} & \dots & D'_{tt} \end{bmatrix} \right). \qquad (12)$$

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Note that $(D_{ji}, D'_{ji}) = (-D_{ij}^T, -D'^T_{ij})$ and define

$$\mathcal{D}(\mathcal{X}_i) := (D_{ii}, D'_{ii}) \quad \text{and} \quad \mathcal{D}(\mathcal{X}_i, \mathcal{X}_j) := (D_{ij}, D'_{ij}), \ i < j. \tag{13}$$

Since each pair of skew-symmetric matrices is congruent to its canonical pair of matrices, it suffices to construct the miniversal deformations for the pairs of canonical matrices (i.e. direct sums of the pairs (1)–(3)).

Theorem 2.1. *Let* $(A, B)_{\mathrm{can}}$ *be a pair of skew-symmetric complex matrices* (4)*. A simplest miniversal deformation of* $(A, B)_{\mathrm{can}}$ *can be taken in the form* $(A, B)_{\mathrm{can}} + \mathcal{D}$ *in which* $\mathcal{D}$ *is a pair of* $(0, *)$ *matrices (the stars denote independent parameters, up to skew-symmetry, see also Remark 2.1) whose matrices are partitioned into blocks conformally to the decomposition of* $(A, B)_{\mathrm{can}}$*, see* (12)*, and the blocks of* $\mathcal{D}$ *are defined, in the notation* (13)*, as follows:*
  (i) *The* diagonal blocks *of* $\mathcal{D}$ *are defined by*

$$\mathcal{D}(\mathcal{H}_n(\lambda)) = \left(0, \begin{bmatrix} 0 & 0^{\swarrow} \\ 0^{\nearrow} & 0 \end{bmatrix}\right), \tag{14}$$

$$\mathcal{D}(\mathcal{K}_n) = \left(\begin{bmatrix} 0 & 0^{\swarrow} \\ 0^{\nearrow} & 0 \end{bmatrix}, 0\right), \tag{15}$$

$$\mathcal{D}(\mathcal{L}_n) = (0, 0). \tag{16}$$

  (ii) *The* off-diagonal blocks *of* $\mathcal{D}$ *whose horizontal and vertical strips contain pairs of* $(A, B)_{\mathrm{can}}$ *of the* same type *are defined by*

$$\mathcal{D}(\mathcal{H}_n(\lambda), \mathcal{H}_m(\mu)) = \begin{cases} (0, 0) & \text{if } \lambda \ne \mu, \\ \left(0, \begin{bmatrix} 0^{\searrow} & 0^{\swarrow} \\ 0^{\nearrow} & 0^{\nwarrow} \end{bmatrix}\right) & \text{if } \lambda = \mu, \end{cases} \tag{17}$$

$$\mathcal{D}(\mathcal{K}_n, \mathcal{K}_m) = \left(\begin{bmatrix} 0^{\searrow} & 0^{\swarrow} \\ 0^{\nearrow} & 0^{\nwarrow} \end{bmatrix}, 0\right), \tag{18}$$

$$\mathcal{D}(\mathcal{L}_n, \mathcal{L}_m) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0_* \end{bmatrix}, \begin{bmatrix} 0 & 0^{\boxminus T}_{m+1,n} \\ 0^{\boxminus}_{n+1,m} & 0^{\urcorner} \end{bmatrix}\right). \tag{19}$$

(iii) *The* off-diagonal blocks *of* $\mathcal{D}$ *whose horizontal and vertical strips contain pairs of* $(A, B)_{\mathrm{can}}$ *of* different types *are defined by*

$$\mathcal{D}(\mathcal{H}_n(\lambda), \mathcal{K}_m) = (0, 0), \tag{20}$$

$$\mathcal{D}(\mathcal{H}_n(\lambda), \mathcal{L}_m) = \left(0, \begin{bmatrix} 0 & 0^{\leftarrow} \end{bmatrix}\right), \tag{21}$$

$$\mathcal{D}(\mathcal{K}_n, \mathcal{L}_m) = \left(0^{\rightarrow}, 0\right). \tag{22}$$

Remark 2.1 (About the independency of parameters). All parameters placed instead of the stars in the upper triangular parts of matrices of $\mathcal{D}$ are independent and the lower triangular parts are defined by the skew-symmetry. In particular, it means that parametric matrix pairs obtained from (D_{ij}, D'_{ij}) and $(D_{i'j'}, D'_{i'j'})$ have dependent (in fact, equal up to the sign) parametric entries if and only if $i' = j$ and $j' = i$.

Let us give an example of how the miniversal deformations from Theorem 2.1 can be used for the investigation of changes of the canonical structure information under small perturbations.

Example 2.1. We show that in an arbitrarily small neighbourhood of a matrix pair with the canonical form $\mathcal{L}_1 \oplus \mathcal{L}_0$ there is always a matrix pair with the canonical form $\mathcal{H}_2(\lambda), \lambda \neq 0$ (in fact, also with $\mathcal{H}_2(0)$ and $\mathcal{K}_2$).

It is enough to consider perturbations of $\mathcal{L}_1 \oplus \mathcal{L}_0$ in the form of the miniversal deformations given in Theorem 2.1 (with only three independent nonzero parameters). Since we will use the theory developed for matrix pencils we switch to the pencil notation $X - sY$, instead of (X, Y). Thus a miniversal deformation of $\mathcal{L}_1 \oplus \mathcal{L}_0$ is the pencil

$$
\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_1 \\ 0 & 0 & -\varepsilon_1 & 0 \end{bmatrix} - s \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \varepsilon_2 \\ -1 & 0 & 0 & \varepsilon_3 \\ 0 & -\varepsilon_2 & -\varepsilon_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -s & 0 \\ -1 & 0 & 0 & -s\varepsilon_2 \\ s & 0 & 0 & \varepsilon_1 - s\varepsilon_3 \\ 0 & s\varepsilon_2 & -\varepsilon_1 + s\varepsilon_3 & 0 \end{bmatrix}, \quad (23)
$$

which has the Smith form (see [31] for the definition)

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \varepsilon_1 - s\varepsilon_2 - s^2\varepsilon_3 & 0 \\ 0 & 0 & 0 & \varepsilon_1 - s\varepsilon_2 - s^2\varepsilon_3 \end{bmatrix}. \quad (24)
$$

In turn, the pencil $\mathcal{H}_2(\lambda)$ is

$$
\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} - s \begin{bmatrix} 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \\ -\lambda & 0 & 0 & 0 \\ -1 & -\lambda & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 - s\lambda & s \\ 0 & 0 & 0 & 1 - s\lambda \\ -1 + s\lambda & 0 & 0 & 0 \\ -s & -1 + s\lambda & 0 & 0 \end{bmatrix}, \quad (25)
$$

and has the Smith form

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 - 2s\lambda - s^2\lambda^2 & 0 \\ 0 & 0 & 0 & 1 - 2s\lambda - s^2\lambda^2 \end{bmatrix}. \quad (26)
$$

Now (24) with $\varepsilon_2 = 2\varepsilon_1\lambda$ and $\varepsilon_3 = \varepsilon_1\lambda^2$ is strictly equivalent to (26) which implies that the pencils (23) and (25) are strictly equivalent by [29, Proposition A.5.1, p. 663] (note that $\lambda \neq 0$ and we must choose $\varepsilon_1 \neq 0$) and due

to [31, Theorem 3, p. 275] the pencils (23) and (25) are congruent. Since ε_1 (and thus ε_2 and ε_3) can be chosen arbitrarily small we can find a pair with the canonical form $\mathcal{H}_2(\lambda), \lambda \neq 0$ in any neighbourhood of $\mathcal{L}_1 \oplus \mathcal{L}_0$.

Note that, for (23) and (25) we could have also computed the skew-symmetric Smith form derived in [33]. The result of this example also follows from the more general result in [16] but the proof given here is constructive, i.e. the perturbation is derived explicitly.

The pair of matrices $\mathcal{D}$ (12) in Theorem 2.1 will be constructed in Section 3 as follows. The vector space

$$T_{(A,B)_{\mathrm{can}}} := \{C^T(A, B)_{\mathrm{can}} + (A, B)_{\mathrm{can}} C \,|\, C \in \mathbb{C}^{\hat{n} \times \hat{n}}\}$$

is the tangent space to the congruence class of $(A, B)_{\mathrm{can}}$ at the point $(A, B)_{\mathrm{can}}$ since

$$(I + \varepsilon C)^T (A, B)_{\mathrm{can}} (I + \varepsilon C) = (A, B)_{\mathrm{can}} + \varepsilon (C^T (A, B)_{\mathrm{can}} + (A, B)_{\mathrm{can}} C) \\ + \varepsilon^2 C^T (A, B)_{\mathrm{can}} C \tag{27}$$

for all $\hat{n}$-by-$\hat{n}$ matrices C and each $\varepsilon \in \mathbb{C}$. Then $\mathcal{D}$ is constructed such that

$$\mathbb{C}_c^{\hat{n} \times \hat{n}} \times \mathbb{C}_c^{\hat{n} \times \hat{n}} = T_{(A,B)_{\mathrm{can}}} + \mathcal{D}(\mathbb{C}) \tag{28}$$

in which $\mathbb{C}_c^{\hat{n} \times \hat{n}}$ is the space of all skew-symmetric $\hat{n} \times \hat{n}$ matrices, $\mathcal{D}(\mathbb{C})$ is the vector space of all pairs of skew-symmetric matrices obtained from $\mathcal{D}$ by replacing its stars by complex numbers, see (6). Thus, one half of the number of stars in $\mathcal{D}$ is equal to the codimension of the congruence orbit of $(A, B)_{\mathrm{can}}$ (note that the total number of the stars is always even). Lemma 3.2 in Section 3.1 ensures that any pair of $(0, *)$ matrices that satisfies (28) can be taken as $\mathcal{D}$ in Theorem 2.1.

2.2. Upper bound for the norm of miniversal deformations

In this section, we bound the distance from the miniversal deformations to a matrix pair that was originally perturbed, using the norm of the perturbations. In particular, we see that this distance can be made arbitrarily small by decreasing the size of the allowed perturbations. Similar techniques are used in [14, 15] to prove the versality of the deformations.

We use the Frobenius norm of a complex $n \times n$ matrix $Y = [y_{ij}]$:

$$\|Y\| := \sqrt{\sum |y_{ij}|^2}.$$

Recall that for matrices Y and Z and $\nu, \omega \in \mathbb{C}$ the following inequalities hold (e.g., see [24, Section 5.6])

$$\|\nu Y + \omega Z\| \leqslant |\nu| \|Y\| + |\omega| \|Z\| \quad \text{and} \quad \|YZ\| \leqslant \|Y\| \|Z\|. \tag{29}$$

Let $(A, B) \in (\mathbb{C}_c^{\hat{n}\times\hat{n}}, \mathbb{C}_c^{\hat{n}\times\hat{n}})$ and $\alpha := \|A\|, \beta := \|B\|$. By (28), for each pair of skew-symmetric $\hat{n}$-by-$\hat{n}$ matrices $(E_{ij}, 0)$ and $(0, E_{i'j'})$, $1 \leqslant i, j, i', j' \leqslant \hat{n}$ there exist $X_{ij}, X'_{i'j'} \in \mathbb{C}^{\hat{n}\times\hat{n}}$ such that

$$
\begin{aligned}
(E_{ij}, 0) + X_{ij}^T(A + M, B + N) + (A + M, B + N)X_{ij} \in \mathcal{D}(\mathbb{C}), \\
(0, E_{i'j'}) + X'^T_{i'j'}(A + M, B + N) + (A + M, B + N)X'_{i'j'} \in \mathcal{D}(\mathbb{C}),
\end{aligned} \tag{30}
$$

where $\mathcal{D}(\mathbb{C})$ is defined in (6). If $(i, j) \in \mathrm{Ind}_1(\mathcal{D})$, then $(E_{ij}, 0) \in \mathcal{D}(\mathbb{C})$, and so we can put $X_{ij} = 0$. Analogously, if $(i', j') \in \mathrm{Ind}_2(\mathcal{D})$, then $(E_{i'j'}, 0) \in \mathcal{D}(\mathbb{C})$, and so we can put $X_{i'j'} = 0$. Denote

$$
\gamma := \sum_{(i,j) \notin \mathrm{Ind}_1(\mathcal{D})} \|X_{ij}\| + \sum_{(i',j') \notin \mathrm{Ind}_2(\mathcal{D})} \|X'_{i'j'}\|. \tag{31}
$$

Theorem 2.2. *Let* $(A, B) \in (\mathbb{C}_c^{\hat{n}\times\hat{n}}, \mathbb{C}_c^{\hat{n}\times\hat{n}})$ *and let* $\varepsilon \in \mathbb{R}$ *such that*

$$
0 < \varepsilon < \frac{1}{\max\{1 + \gamma(\alpha + 1)(2 + \gamma), 1 + \gamma(\beta + 1)(2 + \gamma)\}},
$$

where $\alpha := \|A\|, \beta := \|B\|$ *and* γ *is defined in (31). For each pair of skew-symmetric* $\hat{n}$-*by*-$\hat{n}$ *matrices* (M, N) *satisfying*

$$
\|M\| < \varepsilon^2, \qquad \|N\| < \varepsilon^2, \tag{32}
$$

there exists a matrix $S = I_{\hat{n}} + X$ *depending holomorphically on the entries of* (M, N) *in a neighborhood of zero such that*

$$
S^T(A + M, B + N)S = (A + P, B + Q), \quad (P, Q) \in \mathcal{D}(\mathbb{C}), \ \|P\| < \varepsilon, \text{ and } \|Q\| < \varepsilon,
$$

where $\mathbb{C}_c^{\hat{n}\times\hat{n}} \times \mathbb{C}_c^{\hat{n}\times\hat{n}} = T_{(A,B)_{\mathrm{can}}} + \mathcal{D}(\mathbb{C})$.

Proof. First, note that if $M = 0$ and $N = 0$ then $S = I_{\hat{n}}$.

We construct $S = I_{\hat{n}} + X$. If $M = \sum_{i,j} m_{ij}E_{ij}$ and $N = \sum_{i,j} n_{ij}E_{ij}$ (i.e., $M = [m_{ij}]$ and $N = [n_{ij}]$), then we can chose X_{ij} and X'_{ij} (30), such that

$$
\sum_{i,j}(m_{ij}E_{ij}, n_{ij}E_{ij}) + \sum_{i,j}(m_{ij}X_{ij}^T + n_{ij}X'^T_{ij})(A + M, B + N)
$$
$$
+ (A + M, B + N)\sum_{i,j}(m_{ij}X_{ij} + n_{ij}X'_{ij}) \in \mathcal{D}(\mathbb{C})
$$

and for

$$
X := \sum_{i,j}(m_{ij}X_{ij} + n_{ij}X'_{ij})
$$

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we have

$$(M, N) + X^T(A + M, B + N) + (A + M, B + N)X \in \mathcal{D}(\mathbb{C}).$$

If $(i,j) \notin \mathrm{Ind}_1(\mathcal{D})$ (or, respectively, $(i,j) \notin \mathrm{Ind}_2(\mathcal{D})$), then $|m_{ij}| < \varepsilon^2$ (or, respectively, $|n_{ij}| < \varepsilon^2$) by (32). We obtain

$$\|X\| \leqslant \sum_{(i,j)\notin\mathrm{Ind}_1(\mathcal{D})} |m_{ij}|\|X_{ij}\| + \sum_{(i,j)\notin\mathrm{Ind}_2(\mathcal{D})} |n_{ij}|\|X'_{ij}\|$$
$$< \sum_{(i,j)\notin\mathrm{Ind}_1(\mathcal{D})} \varepsilon^2\|X_{ij}\| + \sum_{(i,j)\notin\mathrm{Ind}_2(\mathcal{D})} \varepsilon^2\|X'_{ij}\| = \varepsilon^2\gamma.$$

Put
$$S^T(A + M, B + N)S = (A + P, B + Q) \quad \text{where } S := I_{\hat{n}} + X,$$

then

$$(P, Q) = (M, N) + X^T(A + M, B + N) + (A + M, B + N)X$$
$$+ X^T(A + M, B + N)X.$$

Summing up, we obtain

$$\|P\| \leqslant \|M\| + 2\|X\|(\|A\| + \|M\|) + \|X\|^2(\|A\| + \|M\|)$$
$$< \varepsilon^2 + 2\varepsilon^2\gamma(\alpha + \varepsilon^2) + \varepsilon^4\gamma^2(\alpha + \varepsilon^2) = \varepsilon^2 + \varepsilon^2\gamma(\alpha + \varepsilon^2)(2 + \varepsilon^2\gamma)$$
$$< \varepsilon^2(1 + \gamma(\alpha + 1)(2 + \gamma)) < \varepsilon,$$
$$\|Q\| \leqslant \|N\| + 2\|X\|(\|B\| + \|N\|) + \|X\|^2(\|B\| + \|N\|)$$
$$< \varepsilon^2(1 + \gamma(\beta + 1)(2 + \gamma)) < \varepsilon.$$

$\square$

3. Proof of the main theorem

3.1. A method of construction of miniversal deformations

We give a method of construction of simplest miniversal deformations, which will be used in the proof of Theorem 2.1.

The deformation (7) is universal in the sense that every deformation $\mathcal{B}(\sigma_1, \ldots, \sigma_l)$ of (A, B) has the form $\mathcal{U}(\vec{\varphi}(\sigma_1, \ldots, \sigma_l))$, where $\varphi_{ij}(\sigma_1, \ldots, \sigma_l)$ are convergent in a neighborhood of $\vec{0}$ power series such that $\varphi_{ij}(\vec{0}) = 0$. Hence every deformation $\mathcal{B}(\sigma_1, \ldots, \sigma_l)$ in Definition 2.1 can be replaced by $\mathcal{U}(\vec{\varepsilon})$, which proves the following lemma.

Lemma 3.1. *The following two conditions are equivalent for any deformation $\mathcal{A}(\delta_1, \ldots, \delta_k)$ of pair of matrices (A, B):*

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(i) *The deformation $\mathcal{A}(\delta_1, \ldots, \delta_k)$ is versal.*

(ii) *The deformation* (7) *is equivalent to* $\mathcal{A}(\varphi_1(\vec{\varepsilon}), \ldots, \varphi_k(\vec{\varepsilon}))$ *in which all $\varphi_i(\vec{\varepsilon})$ are convergent in a neighborhood of $\vec{0}$ power series such that $\varphi_i(\vec{0}) = 0$.*

If U is a subspace of a vector space V, then each set $v + U$ with $v \in V$ is called a *coset of* U *in* V.

Lemma 3.2. *Let $(A, B) \in (\mathbb{C}_c^{\hat{n} \times \hat{n}}, \mathbb{C}_c^{\hat{n} \times \hat{n}})$ and let $\mathcal{D}$ be a pair of $(0, *)$ matrices of the size $\hat{n} \times \hat{n}$. The following are equivalent:*

(i) *The deformation $(A, B) + \mathcal{D}(\vec{\varepsilon})$ defined in* (5) *is miniversal.*

(ii) *The vector space $(\mathbb{C}_c^{\hat{n} \times \hat{n}}, \mathbb{C}_c^{\hat{n} \times \hat{n}})$ decomposes into the sum*

$$(\mathbb{C}_c^{\hat{n} \times \hat{n}}, \mathbb{C}_c^{\hat{n} \times \hat{n}}) = T_{(A,B)} + \mathcal{D}(\mathbb{C}), \quad T_{(A,B)} \cap \mathcal{D}(\mathbb{C}) = \{(A, B)\}. \qquad (33)$$

(iii) *Each coset of $T_{(A,B)}$ in $(\mathbb{C}_c^{\hat{n} \times \hat{n}}, \mathbb{C}_c^{\hat{n} \times \hat{n}})$ contains exactly one matrix pair of the form $\mathcal{D}$.*

Proof. Define the action of the group $GL_{\hat{n}}(\mathbb{C})$ of nonsingular $\hat{n}$-by-$\hat{n}$ matrices on the space $(\mathbb{C}_c^{\hat{n} \times \hat{n}}, \mathbb{C}_c^{\hat{n} \times \hat{n}})$ by

$$(A, B)^S = S^T (A, B) S, \qquad (A, B) \in (\mathbb{C}_c^{\hat{n} \times \hat{n}}, \mathbb{C}_c^{\hat{n} \times \hat{n}}), \quad S \in GL_{\hat{n}}(\mathbb{C}).$$

The orbit $(A, B)^{GL_{\hat{n}}}$ of (A, B) under this action consists of all pairs of skew-symmetric matrices that are congruent to the pair (A, B).

The space $T_{(A,B)}$ is the tangent space to the orbit $(A, B)^{GL_{\hat{n}}}$ at the point (A, B) (see (27)). Hence $\mathcal{D}(\vec{\varepsilon})$ is transversal to the orbit $(A, B)^{GL_{\hat{n}}}$ at the point (A, B) if

$$(\mathbb{C}_c^{\hat{n} \times \hat{n}}, \mathbb{C}_c^{\hat{n} \times \hat{n}}) = T_{(A,B)} + \mathcal{D}(\mathbb{C})$$

(see definitions in [3, § 29]; two subspaces of a vector space are called *transversal* if their sum is equal to the whole space).

This proves the equivalence of (i) and (ii) since a transversal (of the minimal dimension) to the orbit is a (mini)versal deformation [2, Section 1.6]. The equivalence of (ii) and (iii) is obvious. $\qquad \square$

Due to the versality of each deformation $(A, B) + \mathcal{D}(\vec{\varepsilon})$ in which $\mathcal{D}$ satisfies (33): there is a deformation $I(\vec{\varepsilon})$ of the identity matrix such that $(A, B) + \mathcal{D}(\vec{\varepsilon}) = I(\vec{\varepsilon})^T \mathcal{U}(\vec{\varepsilon}) I(\vec{\varepsilon})$, where $\mathcal{U}(\vec{\varepsilon})$ is defined in (7).

Thus, a simplest miniversal deformation of $(A, B) \in (\mathbb{C}_c^{\hat{n} \times \hat{n}}, \mathbb{C}_c^{\hat{n} \times \hat{n}})$ can be constructed as follows. Let $(T_1, \ldots, T_r)$ be a basis of the space $T_{(A,B)}$,

and let $(E_1, \ldots, E_{\hat{n}(\hat{n}-1)})$ be the basis of $(\mathbb{C}_c^{\hat{n}\times\hat{n}}, \mathbb{C}_c^{\hat{n}\times\hat{n}})$ in which every E_k is either of the form $(E_{ij}, 0)$ or $(0, E_{i'j'})$. Removing from the sequence $(T_1, \ldots, T_r, E_1, \ldots, E_{\hat{n}(\hat{n}-1)})$ every pair of matrices that is a linear combination of the preceding matrices, we obtain a new basis $(T_1, \ldots, T_r, E_{i_1}, \ldots, E_{i_k})$ of the space $(\mathbb{C}_c^{\hat{n}\times\hat{n}}, \mathbb{C}_c^{\hat{n}\times\hat{n}})$. By Lemma 3.2, the deformation

$$
\begin{aligned}
\mathcal{A}(\varepsilon_1, \ldots, \varepsilon_{k_1}, \varepsilon_1', \ldots, \varepsilon_{k_2}') &= (A, B) + \varepsilon_1 E_1 + \cdots + \varepsilon_{k_1} E_{i_{k_1}} + \varepsilon_1' E_{i_{k_1+1}} + \cdots + \varepsilon_{k_2}' E_{i_k} \\
&= (A, B) + \varepsilon_1 (E_{i_1, j_1}, 0) + \cdots + \varepsilon_{k_1} (E_{i_{k_1} j_{k_1}}, 0) \\
&\quad + \varepsilon_1' (0, E_{i_{k_1+1}, j_{k_1+1}}) + \cdots + \varepsilon_{k_2}' (0, E_{i_k, j_k}),
\end{aligned}
$$

where $k_1 + k_2 = k$, is miniversal.

For each pair of skew-symmetric $\hat{m} \times \hat{m}$ matrices (A_1, B_1) and each pair of skew-symmetric $\hat{n} \times \hat{n}$ matrices (A_2, B_2), define the vector spaces

$$
V(A_1, B_1) \coloneqq \{S^T(A_1, B_1) + (A_1, B_1)S, \text{ where } S \in \mathbb{C}^{\hat{m}\times\hat{m}}\}, \quad (34)
$$
$$
V((A_1, B_1), (A_2, B_2)) \coloneqq \{(R^T(A_2, B_2) + (A_1, B_1)S, S^T(A_1, B_1) + (A_2, B_2)R), \\
\text{ where } S \in \mathbb{C}^{\hat{m}\times\hat{n}} \text{ and } R \in \mathbb{C}^{\hat{n}\times\hat{m}}\}.
$$
$$(35)$$

Lemma 3.3. *Let $(A, B) = (A_1, B_1) \oplus \cdots \oplus (A_t, B_t)$ be a block-diagonal matrix in which every (A_i, B_i) is $n_i \times n_i$. Let $\mathcal{D}$ be a pair of $(0, *)$ matrices of the size of (A, B). Partitioning $\mathcal{D}$ into blocks (D_{ij}, D_{ij}') conformably to the partitioning of (A, B) (see (12)). Then $(A, B) + \mathcal{D}(E, E')$ is a simplest miniversal (skew-symmetric) deformation of (A, B) under congruence if and only if*

(i) *every coset of $V(A_i, B_i)$ in $(\mathbb{C}_c^{n_i\times n_i}, \mathbb{C}_c^{n_i\times n_i})$ contains exactly one matrix of the form (D_{ii}, D_{ii}'), and*

(ii) *every coset of $V((A_i, B_i), (A_j, B_j))$ in $(\mathbb{C}^{n_i\times n_j}, \mathbb{C}^{n_i\times n_j}) \oplus (\mathbb{C}^{n_j\times n_i}, \mathbb{C}^{n_j\times n_i})$ contains exactly two pairs of matrices $((W_1, W_2), (-W_1^T, -W_2^T))$ in which (W_1, W_2) is of the form (D_{ij}, D_{ij}') and correspondingly $(-W_1^T, -W_2^T)$ is of the form $(D_{ji}, D_{ji}') = (-D_{ij}^T, -D_{ij}'^T)$.*

Proof. By Lemma 3.2(iii), $(A, B) + \mathcal{D}(\vec{\varepsilon})$ is a simplest miniversal deformation of (A, B) if and only if for each $(C, C') \in (\mathbb{C}_c^{\hat{n}\times\hat{n}}, \mathbb{C}_c^{\hat{n}\times\hat{n}})$ the coset $(C, C') + T_{(A,B)}$ contains exactly one (D, D') of the form $\mathcal{D}$, that is,

$$
(D, D') = (C, C') + S^T(A, B) + (A, B)S \in \mathcal{D}(\mathbb{C}) \qquad \text{with } S \in \mathbb{C}^{\hat{n}\times\hat{n}}. \quad (36)
$$

Partition (D, D'), (C, C'), and S into blocks conformably to the partitioning of (A, B). By (36), for each i we have $(D_{ii}, D_{ii}') = (C_{ii}, C_{ii}') + S_{ii}^T(A_i, B_i) +$

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$(A_i, B_i)S_{ii}$, and for all i and j such that $i < j$ we have

$$
\left(\begin{bmatrix} D_{ii} & D_{ij} \\ D_{ji} & D_{jj} \end{bmatrix}, \begin{bmatrix} D'_{ii} & D'_{ij} \\ D'_{ji} & D'_{jj} \end{bmatrix} \right) = \left(\begin{bmatrix} C_{ii} & C_{ij} \\ C_{ji} & C_{jj} \end{bmatrix}, \begin{bmatrix} C'_{ii} & C'_{ij} \\ C'_{ji} & C'_{jj} \end{bmatrix} \right)
$$
$$
+ \begin{bmatrix} S_{ii}^T & S_{ji}^T \\ S_{ij}^T & S_{jj}^T \end{bmatrix} \left(\begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix}, \begin{bmatrix} B_i & 0 \\ 0 & B_j \end{bmatrix} \right) + \left(\begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix}, \begin{bmatrix} B_i & 0 \\ 0 & B_j \end{bmatrix} \right) \begin{bmatrix} S_{ii} & S_{ij} \\ S_{ji} & S_{jj} \end{bmatrix}. \quad (37)
$$

Thus, (36) is equivalent to the conditions

$$
(D_{ii}, D'_{ii}) = (C_{ii}, C'_{ii}) + S_{ii}^T(A_i, B_i) + (A_i, B_i)S_{ii} \in \mathcal{D}_{ii}(\mathbb{C}), 1 \leqslant i \leqslant t, \quad (38)
$$

$$
((D_{ij}, D'_{ij}), (D_{ji}, D'_{ji})) = ((C_{ij}, C'_{ij}), (C_{ji}, C'_{ji}))
$$
$$
+ ((S_{ji}^T A_j + A_i S_{ij}, S_{ji}^T B_j + B_i S_{ij}), (S_{ij}^T A_i + A_j S_{ji}, S_{ij}^T B_i + B_j S_{ji}))
$$
$$
\in \mathcal{D}_{ij}(\mathbb{C}) \oplus \mathcal{D}_{ji}(\mathbb{C}), \quad 1 \leqslant i < j \leqslant t. \quad (39)
$$

Hence for each $(C, C') \in (\mathbb{C}_c^{\hat{n} \times \hat{n}}, \mathbb{C}_c^{\hat{n} \times \hat{n}})$ there exists exactly one $(D, D') \in \mathcal{D}(\mathbb{C})$ of the form (36) if and only if

(i′) for each $(C_{ii}, C'_{ii}) \in (\mathbb{C}_c^{n_i \times n_i}, \mathbb{C}_c^{n_i \times n_i})$ there exists exactly one $(D_{ii}, D'_{ii}) \in \mathcal{D}_{ii}(\mathbb{C})$ of the form (38), and

(ii′) for each $((C_{ij}, C'_{ij}), (C_{ji}, C'_{ji})) \in (\mathbb{C}^{n_i \times n_j}, \mathbb{C}^{n_i \times n_j}) \oplus (\mathbb{C}^{n_j \times n_i}, \mathbb{C}^{n_j \times n_i})$ there exists exactly one $((D_{ij}, D'_{ij}), (D_{ji}, D'_{ji})) \in \mathcal{D}_{ij}(\mathbb{C}) \oplus \mathcal{D}_{ji}(\mathbb{C})$ of the form (39).

This proves the lemma. □

Corollary 3.1. *In the notation of Lemma 3.3, $(A, B) + \mathcal{D}(\vec{\varepsilon})$ is a miniversal deformation of (A, B) if and only if each pair of submatrices of the form*

$$
\left(\begin{bmatrix} A_i + D_{ii}(\vec{\varepsilon}) & D_{ij}(\vec{\varepsilon}) \\ D_{ji}(\vec{\varepsilon}) & A_j + D_{jj}(\vec{\varepsilon}) \end{bmatrix} \begin{bmatrix} B_i + D'_{ii}(\vec{\varepsilon}) & D'_{ij}(\vec{\varepsilon}) \\ D'_{ji}(\vec{\varepsilon}) & B_j + D'_{jj}(\vec{\varepsilon}) \end{bmatrix} \right) \quad \text{with } i < j,
$$

is a miniversal deformation of the pair $(A_i \oplus A_j, B_i \oplus B_j)$.

We are ready to prove Theorem 2.1 now. Each $\mathcal{X}_i$ in (11) is of the form $\mathcal{H}_n(\lambda), \mathcal{K}_n$, or $\mathcal{L}_n$, and so there are 9 types of pairs $\mathcal{D}(\mathcal{X}_i)$ and $\mathcal{D}(\mathcal{X}_i, \mathcal{X}_j)$ with $i < j$; they are given in (14)–(22). It suffices to prove that the pairs (14)–(22) satisfy the conditions (i) and (ii) of Lemma 3.3.

3.2. Diagonal blocks of $\mathcal{D}$

Fist we verify that the diagonal blocks of $\mathcal{D}$ defined in part (i) of Theorem 2.1 satisfy the condition (i) of Lemma 3.3.

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3.2.1. Diagonal blocks $\mathcal{D}(\mathcal{H}_n(\lambda))$ and $\mathcal{D}(\mathcal{K}_n)$

We consider the pairs of blocks $\mathcal{H}_n(\lambda)$ and $\mathcal{K}_n$.

Due to Lemma 3.3(i), it suffices to prove that each pair of skew-symmetric $2n$-by-$2n$ matrices $(A, B) = ([A_{ij}]^2_{i,j=1}, [B_{ij}]^2_{i,j=1})$ can be reduced to exactly one pair of matrices of the form (14) by adding

$$
\begin{aligned}
\Delta(A, B) = (\Delta A, \Delta B) &= \left(\begin{bmatrix} \Delta A_{11} & \Delta A_{12} \\ \Delta A_{21} & \Delta A_{22} \end{bmatrix}, \begin{bmatrix} \Delta B_{11} & \Delta B_{12} \\ \Delta B_{21} & \Delta B_{22} \end{bmatrix} \right) \\
&= \begin{bmatrix} S_{11}^T & S_{21}^T \\ S_{12}^T & S_{22}^T \end{bmatrix} \left(\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} \right) \\
&\quad + \left(\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} \right) \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \\
&= \left(\begin{bmatrix} S_{21} - S_{21}^T & S_{11}^T + S_{22} \\ -S_{11} - S_{22}^T & S_{12}^T - S_{12} \end{bmatrix}, \right. \\
&\quad \left. \begin{bmatrix} -S_{21}^T J_n(\lambda)^T + J_n(\lambda)S_{21} & S_{11}^T J_n(\lambda) + J_n(\lambda)S_{22} \\ -S_{22}^T J_n(\lambda)^T - J_n(\lambda)^T S_{11} & S_{12}^T J_n(\lambda) - J_n(\lambda)^T S_{12} \end{bmatrix} \right),
\end{aligned}
\tag{40}
$$

in which $S = [S_{ij}]^2_{i,j=1}$ is an arbitrary $2n$-by-$2n$ matrix. Due to the skew-symmetry there are three pairs of n-by-n blocks in (40) that can be treated independently. For any X we have

$$
-X J_n(\lambda)^T + J_n(\lambda)X = -X(\lambda I + J_n(0))^T + (\lambda I + J_n(0))X = -X J_n(0)^T + J_n(0)X.
$$

Thus, without loss of generality, we can assume that $\lambda = 0$. Therefore the deformation of $\mathcal{K}_n$ is equal to the deformation of $\mathcal{H}_n(\lambda)$ up to the permutation of matrices.

First we consider the pair of blocks $\Delta(A_{11}, B_{11}) = (S_{21} - S_{21}^T, -S_{21}^T J_n(0)^T + J_n(0)S_{21})$ in which S_{21} is an arbitrary n-by-n matrix. Obviously, by adding $\Delta A_{11} = S_{21} - S_{21}^T$ we reduce A_{11} to zero. To preserve A_{11}, we must hereafter take S_{21} such that $S_{21} - S_{21}^T = 0$, i.e., S_{21} is symmetric. We reduce B_{11} by adding $\Delta B_{11} = -S_{21}^T J_n(0)^T + J_n(0)S_{21}$,

$\Delta B_{11} =$

$$
= - \begin{bmatrix} s_{11} & s_{12} & s_{13} & \cdots & s_{1n} \\ s_{12} & s_{22} & s_{23} & \cdots & s_{2n} \\ s_{13} & s_{23} & s_{33} & \cdots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \cdots & s_{nn} \end{bmatrix} \begin{bmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} & s_{13} & \cdots & s_{1n} \\ s_{12} & s_{22} & s_{23} & \cdots & s_{2n} \\ s_{13} & s_{23} & s_{33} & \cdots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \cdots & s_{nn} \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & s_{22} - s_{13} & s_{23} - s_{14} & \cdots & s_{2n} \\ -s_{22} + s_{13} & 0 & s_{33} - s_{24} & \cdots & s_{3n} \\ -s_{23} + s_{14} & -s_{33} + s_{24} & 0 & \cdots & s_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -s_{2n} & -s_{3n} & -s_{4n} & \cdots & 0 \end{bmatrix}.
$$

$$
\tag{41}
$$

We reduce B_{11} anti-diagonal-wise and since B_{11} is skew-symmetric, we just need to reduce the upper triangular part of B_{11} and the lower triangular part will be reduced automatically. Let $b = (b_1, \ldots, b_{t-1})$ denote the elements of the upper half of the k-th anti-diagonal (counting from the top left corner) of B_{11}. Each of the first $(n-1)$ upper halves of the anti-diagonals of ΔB_{11} is of the form

$$s = \begin{cases} (s_{2k} - s_{1,k+1}, s_{3,k-1} - s_{2k}, \ldots, s_{tt} - s_{t-1,t+1}), & \text{if k is even, } t = \frac{k+2}{2}; \\ (s_{2k} - s_{1,k+1}, s_{3,k-1} - s_{2k}, \ldots, s_{t,t+1} - s_{t-1,t+2}), & \text{if k is odd, } t = \frac{k+1}{2}, \end{cases}$$

where $k = 2, 3, \ldots, n - 1$, (the first anti-diagonal is zero). Choosing the parameters s_{ij} we want to make s equal to b, i.e. we want to solve the system of linear equations

$$\begin{bmatrix} -1 & 1 & & & 0 \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & -1 & 1 \end{bmatrix} \begin{bmatrix} s_{1,k+1} \\ s_{2k} \\ \vdots \\ s_{tt} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{t-1} \end{bmatrix}, \tag{42}$$

where k is even (and the analogous system for k being odd). The system (42) has a solution. Therefore, we can reduce each of the first $(n-1)$ anti-diagonals of B_{11} to zero, by adding the corresponding anti-diagonals of ΔB_{11}.

For each k-th upper parts of the last n anti-diagonals we have the following systems of equations

$$\begin{bmatrix} 1 & & & & 0 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & -1 & 1 \end{bmatrix} \begin{bmatrix} s_{2-n+k,n} \\ s_{3-n+k,n-1} \\ \vdots \\ s_{t'-1,t'+1} \\ s_{t't'} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{t'-2} \\ b_{t'-1} \end{bmatrix}, \tag{43}$$

where $k = n, n + 1, \ldots, 2n - 2$, (the last anti-diagonal is zero) and $t' = t - k + n$ and t is defined as above. The system (43) has a solution. Therefore we can reduce the last n anti-diagonals of B_{11} to zero. Altogether, we reduce B_{11} to zero matrix by adding ΔB_{11}.

The possibility of reducing (A_{22}, B_{22}) to zero by adding $\Delta(A_{22}, B_{22}) = (S_{12}^T - S_{12}, S_{12}^T J_n(0) - J_n(0)^T S_{12})$ follows directly from the reduction of the blocks (A_{11}, B_{11}). We have $0 = B_{11} - S_{21}^T J_n(0)^T + J_n(0) S_{21}$ where B_{11} is a skew-symmetric matrix. Multiplying this equality by the n-by-n flip matrix

$$Z := \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix} \tag{44}$$

from both sides and using that $Z^2 = I$ and $ZJ_n(0)^T Z = J_n(0)$ we get

$$0 = ZB_{11}Z - ZS_{21}^T ZJ_n(0) + J_n(0)^T ZS_{21}Z.$$

This ensures that the pair of blocks (A_{22}, B_{22}) can be set to zero since $ZB_{11}Z$ and $ZS_{21}Z$ are arbitrary skew-symmetric and symmetric matrices, respectively.

To the pair of blocks (A_{21}, B_{21}) we can add $\Delta(A_{21}, B_{21}) = (S_{11}^T + S_{22}, S_{11}^T J_n(0) + J_n(0)S_{22})$. Adding $S_{11}^T + S_{22}$ we reduce A_{21} to zero. To preserve A_{21}, we must hereafter take S_{11} and S_{22} such that $S_{11}^T = -S_{22}$. Thus we add $\Delta B_{21} = -S_{22}J_n(0) + J_n(0)S_{22}$, with any matrix S_{22},

$\Delta B_{21} =$

$$= -\begin{bmatrix} s_{11} & s_{12} & s_{13} & \cdots & s_{1n} \\ s_{21} & s_{22} & s_{23} & \cdots & s_{2n} \\ s_{31} & s_{32} & s_{33} & \cdots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & s_{n3} & \cdots & s_{nn} \end{bmatrix} \begin{bmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & 1 & \\ & & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & 1 & \\ & & & & 0 \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} & s_{13} & \cdots & s_{1n} \\ s_{21} & s_{22} & s_{23} & \cdots & s_{2n} \\ s_{31} & s_{32} & s_{33} & \cdots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & s_{n3} & \cdots & s_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} s_{21} & s_{22} - s_{11} & s_{23} - s_{12} & \cdots & s_{2n} - s_{1,n-1} \\ s_{31} & s_{32} - s_{21} & s_{33} - s_{22} & \cdots & s_{3n} - s_{2,n-1} \\ s_{41} & s_{42} - s_{31} & s_{43} - s_{32} & \cdots & s_{4n} - s_{3,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -s_{n1} & -s_{n2} & \cdots & -s_{n,n-1} \end{bmatrix}.$$

(45)

We examine each diagonal of ΔB_{21} independently since each diagonal has unique variables. For each of the first n diagonals (starting from the bottom left corner) we have the following system of equations

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ & & & -1 \end{bmatrix} \begin{bmatrix} s_{n+2-k,1} \\ \vdots \\ s_{n,k-1} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix}.$$

(46)

The matrix of this system has $k-1$ columns and k (since the first diagonal is zero $k = 2, \ldots, n$) rows and its rank is equal to $k-1$ but the rank of the full matrix of the system is k; by the Kronecker-Capelli theorem [26] the system (46) does not have a solution. Nevertheless, if we turn down the first or the last equation of the system (i.e. we do not set the first or the last element of the corresponding diagonal of B_{21} to zero), then (46) will have a solution.

For the last $(n-1)$ diagonals we have a system of equations like (42), which has a solution. Therefore we can set each element of the matrix B_{21} to zero except the elements either in the first column or the last row.

The blocks $\Delta(A_{12}, B_{12}) = (-S_{11} - S_{22}^T, -S_{22}^T J_n(0)^T - J_n(0)^T S_{11})$ are equal to $\Delta(A_{21}, B_{21})$ up to the transposition and sign.

Altogether, we obtain

$$\mathcal{D}(\mathcal{H}_n(\lambda)) = \left(0, \begin{bmatrix} 0 & 0^{\swarrow} \\ 0^{\nearrow} & 0 \end{bmatrix} \right) \quad \text{and} \quad \mathcal{D}(\mathcal{K}_n) = \left(\begin{bmatrix} 0 & 0^{\swarrow} \\ 0^{\nearrow} & 0 \end{bmatrix}, 0 \right).$$

3.2.2. Diagonal blocks $\mathcal{D}(L_n)$

Using Lemma 3.3(i), like in Section 3.2.1, we prove that each pair $(A, B) = ([A_{ij}]^2_{i,j=1}, [B_{ij}]^2_{i,j=1})$ of skew-symmetric $(2n+1)$-by-$(2n+1)$ matrices can be set to zero by adding

$$
\begin{aligned}
\Delta(A, B) &= (\Delta A, \Delta B) = \left(\begin{bmatrix} \Delta A_{11} & \Delta A_{12} \\ \Delta A_{21} & \Delta A_{22} \end{bmatrix}, \begin{bmatrix} \Delta B_{11} & \Delta B_{12} \\ \Delta B_{21} & \Delta B_{22} \end{bmatrix} \right) \\
&= \begin{bmatrix} S^T_{11} & S^T_{21} \\ S^T_{12} & S^T_{22} \end{bmatrix} \left(\begin{bmatrix} 0 & F_n \\ -F^T_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & G_n \\ -G^T_n & 0 \end{bmatrix} \right) + \left(\begin{bmatrix} 0 & F_n \\ -F^T_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & G_n \\ -G^T_n & 0 \end{bmatrix} \right) \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \\
&= \left(\begin{bmatrix} -S^T_{21}F^T_n + F_n S_{21} & S^T_{11}F_n + F_n S_{22} \\ -S^T_{22}F^T_n - F^T_n S_{11} & S^T_{12}F_n - F^T_n S_{12} \end{bmatrix}, \begin{bmatrix} -S^T_{21}G^T_n + G_n S_{21} & S^T_{11}G_n + G_n S_{22} \\ -S^T_{22}G^T_n - G^T_n S_{11} & S^T_{12}G_n - G^T_n S_{12} \end{bmatrix} \right),
\end{aligned}
\tag{47}
$$

where $S = [S_{ij}]^2_{i,j=1}$ is an arbitrary matrix. Each pair of blocks $(A_{ij}, B_{ij}), i, j = 1, 2$, of (A, B) is changed independently.

We add $\Delta(A_{11}, B_{11}) = (-S^T_{21}F^T_n + F_n S_{21}, -S^T_{21}G^T_n + G_n S_{21})$ in which S_{21} is an arbitrary $(n+1)$-by-n matrix to the pair of blocks (A_{11}, B_{11}). Obviously, by adding $-S^T_{21}F^T_n + F_n S_{21}$ we reduce A_{11} to zero. To preserve A_{11}, we must hereafter take S_{21} such that $F_n S_{21} = S^T_{21}F^T_n$. Thus S_{21} without the last row is $n \times n$ and symmetric:

$$
S_{21} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & \cdots & s_{1n} \\ s_{12} & s_{22} & s_{23} & \cdots & s_{2n} \\ s_{13} & s_{23} & s_{33} & \cdots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \cdots & s_{nn} \\ s_{1,n+1} & s_{2,n+1} & s_{3,n+1} & \cdots & s_{n,n+1} \end{bmatrix}.
$$

Now we reduce B_{11} by adding

$$\Delta B_{11} = - \begin{bmatrix} s_{11} & s_{12} & s_{13} & \cdots & s_{1n} & s_{1,n+1} \\ s_{12} & s_{22} & s_{23} & \cdots & s_{2n} & s_{2,n+1} \\ s_{13} & s_{23} & s_{33} & \cdots & s_{3n} & s_{3,n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \cdots & s_{nn} & s_{n,n+1} \end{bmatrix} \begin{bmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & & 0 \\ 0 & & & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} & s_{13} & \cdots & s_{1n} \\ s_{12} & s_{22} & s_{23} & \cdots & s_{2n} \\ s_{13} & s_{23} & s_{33} & \cdots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \cdots & s_{nn} \\ s_{1,n+1} & s_{2,n+1} & s_{3,n+1} & \cdots & s_{n,n+1} \end{bmatrix}$$

$$= \begin{cases} -s_{i,j+1} + s_{i+1,j} & \text{if} \quad i < j, \\ s_{i,j+1} - s_{i+1,j} & \text{if} \quad i > j, \\ 0 & \text{if} \quad i = j, \end{cases} \qquad (48)$$

where $i, j = 1, \ldots, n$. The upper part of each anti-diagonal of ΔB_{11} has unique variables. Thus we reduce each anti-diagonal of B_{11} independently. We have a system of equations (42) for the upper part of each anti-diagonal, which has a solution. It follows that we can reduce every anti-diagonal of B_{11} to zero. Hence we can reduce (A_{11}, B_{11}) to zero by adding $\Delta(A_{11}, B_{11})$.

To the pair of blocks (A_{12}, B_{12}) we can add $\Delta(A_{12}, B_{12}) = (S_{11}^T F_n + F_n S_{22}, S_{11}^T G_n + G_n S_{22})$ in which S_{11} and S_{22} are arbitrary matrices of corresponding size. Adding $S_{11}^T F_n + F_n S_{22}$, we reduce A_{12} to zero. To preserve A_{12}, we must hereafter take S_{11} and S_{22} such that $F_n S_{22} = -S_{11}^T F_n$. This means that

$$S_{22} = \begin{bmatrix} & & & 0 \\ & -S_{11}^T & & 0 \\ & & & \vdots \\ & & & 0 \\ y_1 & y_2 & \cdots & y_{n+1} \end{bmatrix}.$$

Therefore we reduce B_{12} by adding

$$\Delta B_{12} = S_{11}^T G_n + G_n S_{22} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & \dots & s_{1n} \\ s_{21} & s_{22} & s_{23} & \dots & s_{2n} \\ s_{31} & s_{32} & s_{33} & \dots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & s_{n3} & \dots & s_{nn} \end{bmatrix} \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_{11} & -s_{12} & \dots & -s_{1n} & 0 \\ -s_{21} & -s_{22} & \dots & -s_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -s_{n1} & -s_{n2} & \dots & -s_{nn} & 0 \\ y_1 & y_2 & \dots & y_n & y_{n+1} \end{bmatrix}$$

$$= - \begin{bmatrix} s_{21} & -s_{11} + s_{22} & -s_{12} + s_{23} & \dots & -s_{1,n-1} + s_{2n} & -s_{1n} \\ s_{31} & -s_{21} + s_{32} & -s_{22} + s_{33} & \dots & -s_{2,n-1} + s_{3n} & -s_{2n} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ s_{n1} & -s_{n-1,1} + s_{n2} & -s_{n-1,2} + s_{n3} & \dots & -s_{n,n-1} + s_{nn} & -s_{n-1n} \\ -y_1 & -s_{n1} - y_2 & -s_{n2} - y_3 & \dots & -s_{n,n-1} - y_n & -s_{nn} - y_{n+1} \end{bmatrix}. \quad (49)$$

It is easily seen that we can set B_{12} to zero by adding ΔB_{12} (diagonal-wise).

The pair of blocks $\Delta(A_{21}, B_{21}) = (-S_{22}^T F_n^T - F_n^T S_{11}, -S_{22}^T G_n^T - G_n^T S_{11})$ is analogous to $\Delta(A_{12}, B_{12})$ up to transposition and sign.

To the pair of blocks (A_{22}, B_{22}) we add $\Delta(A_{22}, B_{22}) = (S_{12}^T F_n - F_n^T S_{12}, S_{12}^T G_n - G_n^T S_{12})$ in which S_{12} is an arbitrary n-by-$(n+1)$ matrix. Obviously, by adding $S_{12}^T F_n - F_n^T S_{12}$, we reduce A_{22} to zero. To preserve A_{22}, we must hereafter take S_{12} such that $S_{12}^T F_n = F_n^T S_{12}$. Thus

$$S_{12} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & \dots & s_{1n} & 0 \\ s_{12} & s_{22} & s_{23} & \dots & s_{2n} & 0 \\ s_{13} & s_{23} & s_{33} & \dots & s_{3n} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \dots & s_{nn} & 0 \end{bmatrix}.$$

The matrix S_{12} without the last column is $n \times n$ and symmetric. Now we

reduce B_{22} by adding

$\Delta B_{22} =$

$$
= \begin{bmatrix}
s_{11} & s_{12} & s_{13} & \cdots & s_{1n} \\
s_{12} & s_{22} & s_{23} & \cdots & s_{2n} \\
s_{13} & s_{23} & s_{33} & \cdots & s_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
s_{1n} & s_{2n} & s_{3n} & \cdots & s_{nn} \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & & 0 \\
& \ddots & \ddots & \\
0 & & 0 & 1
\end{bmatrix}
- \begin{bmatrix}
0 & & & 0 \\
1 & \ddots & & \\
& \ddots & \ddots & 0 \\
0 & & 1
\end{bmatrix}
\begin{bmatrix}
s_{11} & s_{12} & s_{13} & \cdots & s_{1n} & 0 \\
s_{12} & s_{22} & s_{23} & \cdots & s_{2n} & 0 \\
s_{13} & s_{23} & s_{33} & \cdots & s_{3n} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
s_{1n} & s_{2n} & s_{3n} & \cdots & s_{nn} & 0
\end{bmatrix}
$$

$$
= \begin{bmatrix}
0 & s_{11} & s_{12} & s_{13} & \cdots & s_{1,n-1} & s_{1n} \\
-s_{11} & 0 & s_{22}-s_{13} & s_{23}-s_{14} & \cdots & s_{2,n-1}-s_{1n} & s_{2n} \\
-s_{12} & s_{13}-s_{22} & 0 & s_{33}-s_{24} & \cdots & s_{3,n-1}-s_{2n} & s_{3n} \\
-s_{13} & s_{14}-s_{23} & s_{24}-s_{33} & 0 & \cdots & s_{4,n-1}-s_{3n} & s_{4n} \\
\hdotsfor{7} \\
-s_{1,n-1} & s_{1n}-s_{2,n-1} & s_{2n}-s_{3,n-1} & s_{3n}-s_{4,n-1} & \cdots & 0 & s_{nn} \\
-s_{1n} & -s_{2n} & -s_{3n} & -s_{4n} & \cdots & -s_{nn} & 0
\end{bmatrix}.
$$

We have a system of equations of type (43) which has a solution for the upper part of each anti-diagonal. It follows that we can reduce every anti-diagonal of B_{22} to zero. Hence we can reduce (A_{22}, B_{22}) to zero by adding $\Delta(A_{22}, B_{22})$.

Summing up the analysis for all pairs of blocks, we get $\mathcal{D}(L_n) = 0$.

3.3. Off-diagonal blocks of $\mathcal{D}$ that correspond to summands of $(A, B)_{can}$ of the same type

Now we verify the condition (ii) of Lemma 3.3 for off-diagonal blocks of $\mathcal{D}$ defined in Theorem 2.1(ii); the diagonal blocks of their horizontal and vertical strips contain summands of $(A, B)_{\text{can}}$ of the same type.

3.3.1. Pairs of blocks $\mathcal{D}(\mathcal{H}_n(\lambda), \mathcal{H}_m(\mu))$ and $\mathcal{D}(\mathcal{K}_n, \mathcal{K}_m)$

Due to Lemma 3.3(ii), it suffices to prove that each group of four matrices $((A, B), (-A^T, -B^T))$ can be reduced to exactly one group of the form (17) by adding

$$(R^T \mathcal{H}_m(\mu) + \mathcal{H}_n(\lambda) S, S^T \mathcal{H}_n(\lambda) + \mathcal{H}_m(\mu)) R), \quad S \in \mathbb{C}^{2n \times 2m}, R \in \mathbb{C}^{2m \times 2n}.$$

Obviously, if we reduce the first pair of matrices, the second pair will be reduced automatically. So we reduce a pair (A, B) of $2n$-by-$2m$ matrices by adding

$$\Delta(A, B) = R^T \mathcal{H}_m(\mu) + \mathcal{H}_n(\lambda) S =$$
$$\left(R^T \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S, R^T \begin{bmatrix} 0 & J_m(\mu) \\ -J_m(\mu)^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} S \right).$$

It is clear that we can reduce A to zero. To preserve A, we must hereafter choose $R = [R_{ij}]_{i,j=1}^2$ and $S = [S_{ij}]_{i,j=1}^2$ such that

$$R^T \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S = 0, \text{ or equivalently } \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} -R_{22}^T & R_{12}^T \\ R_{21}^T & -R_{11}^T \end{bmatrix}.$$

Now $B := [B_{ij}]_{i,j=1}^2$ is reduced by adding

$$\begin{aligned}
\Delta B &:= \begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} 0 & J_m(\mu) \\ -J_m(\mu)^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} \begin{bmatrix} -R_{22}^T & R_{12}^T \\ R_{21}^T & -R_{11}^T \end{bmatrix} \\
&= \begin{bmatrix} -R_{21}^T J_m(\mu)^T + J_n(\lambda) R_{21}^T & R_{11}^T J_m(\mu) - J_n(\lambda) R_{11}^T \\ -R_{22}^T J_m(\mu)^T + J_n(\lambda)^T R_{22}^T & R_{12}^T J_m(\mu) - J_n(\lambda)^T R_{12}^T \end{bmatrix}.
\end{aligned}$$

Therefore B_{11} is reduced by adding

$$\begin{aligned}
\Delta B_{11} &= -R_{21}^T J_m(\mu)^T + J_n(\lambda) R_{21}^T \\
&= \begin{cases} (\lambda - \mu) r_{ij} + r_{i+1,j} - r_{i,j+1} & \text{if } 1 \leq i \leq (n-1), 1 \leq j \leq (m-1), \\ (\lambda - \mu) r_{ij} + r_{i+1,j} & \text{if } 1 \leq i \leq (n-1), j = m, \\ (\lambda - \mu) r_{ij} - r_{i,j+1} & \text{if } 1 \leq j \leq (m-1), i = n, \\ (\lambda - \mu) r_{ij} & \text{if } i = n, j = m. \end{cases}
\end{aligned}$$

We have a system of nm equations which has a solution if $\lambda \neq \mu$. Thus for $\lambda \neq \mu$ we can set B_{11} to zero by adding ΔB_{11}.

Now we consider $\lambda = \mu$, i.e.

$$\Delta B_{11} = -R_{21}^T J_m(\lambda)^T + J_n(\lambda) R_{21}^T$$

$$= \begin{bmatrix}
r_{21} - r_{12} & r_{22} - r_{13} & r_{23} - r_{14} & \cdots & r_{2,m-1} - r_{1m} & r_{2m} \\
r_{31} - r_{22} & r_{32} - r_{23} & r_{33} - r_{24} & \cdots & r_{3,m-1} - r_{2m} & r_{3m} \\
r_{41} - r_{32} & r_{42} - r_{33} & r_{43} - r_{34} & \cdots & r_{4,m-1} - r_{3m} & r_{4m} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
r_{n1} - r_{n-1,2} & r_{n2} - r_{n-1,3} & r_{n3} - r_{n-1,4} & \cdots & r_{n,m-1} - r_{n-1,m} & r_{nm} \\
-r_{n2} & -r_{n3} & -r_{n4} & \cdots & -r_{nm} & 0
\end{bmatrix}.$$

Like for the system (45), B_{11} can be reduced to $0^\searrow$ by adding ΔB_{11}.

To find the solutions for the other cases we need to multiply the answer

for the block B_{11} by $\pm Z$:

$$B_{12} - R_{11}^T J_m(\mu) - J_n(\lambda) R_{11}^T$$
$$= B_{11}Z + R_{21}^T ZZ J_m(\mu)^T Z - J_n(\lambda) R_{21}^T Z = \begin{cases} 0Z = 0 & \text{if } \lambda \neq \mu, \\ 0 \searrow Z = 0 \swarrow & \text{if } \lambda = \mu, \end{cases}$$

$$B_{21} - R_{22}^T J_m(\mu)^T + J_n(\lambda)^T R_{22}^T$$
$$= ZB_{11} + ZR_{21}^T J_m(\mu)^T - ZJ_n(\lambda) ZZ R_{21}^T = \begin{cases} Z0 = 0 & \text{if } \lambda \neq \mu, \\ Z0 \searrow = 0 \nearrow & \text{if } \lambda = \mu, \end{cases}$$

$$B_{22} - R_{12}^T J_m(\mu) - J_n(\lambda)^T R_{12}^T$$
$$= ZB_{11}Z + ZR_{21}^T ZZ J_m(\mu)^T Z - ZJ_n(\lambda) ZZ R_{21}^T Z = \begin{cases} Z0Z = 0 & \text{if } \lambda \neq \mu, \\ Z0 \searrow Z = 0 \nwarrow & \text{if } \lambda = \mu. \end{cases}$$

Summing up the derivations for all blocks, we get that $\mathcal{D}(\mathcal{H}_n(\lambda), \mathcal{H}_m(\mu))$ is equal to (17) and, respectively, $\mathcal{D}(\mathcal{K}_n, \mathcal{K}_m)$ is equal to (18).

3.3.2. Pairs of blocks $\mathcal{D}(\mathcal{L}_n, \mathcal{L}_m)$

Due to Lemma 3.3(ii), it suffices to prove that each group of four matrices $((A, B), (-A^T, -B^T))$ can be reduced to exactly one group of the form (19) by adding

$$(R^T \mathcal{L}_m + \mathcal{L}_n S, S^T \mathcal{L}_n + \mathcal{L}_m R), \quad S \in \mathbb{C}^{2n+1 \times 2m+1}, \ R \in \mathbb{C}^{2m+1 \times 2n+1}.$$

It is enough to reduce only the first pair of matrices, i.e. (A, B). We reduce it by adding

$$\Delta(A, B) = R^T \mathcal{L}_m + \mathcal{L}_n S$$
$$= \left(R^T \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & F_n \\ -F_n^T & 0 \end{bmatrix} S, R^T \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & G_n \\ -G_n^T & 0 \end{bmatrix} S \right).$$

It is easily seen that we can set A to zero. To preserve A, we must hereafter take $R = [R_{ij}]_{i,j=1}^2$ and $S = [S_{ij}]_{i,j=1}^2$ such that

$$\begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & F_n \\ -F_n^T & 0 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = 0,$$

or equivalently

$$\begin{bmatrix} -R_{21}^T F_m^T & R_{11}^T F_m \\ -R_{22}^T F_m^T & R_{12}^T F_m \end{bmatrix} = \begin{bmatrix} -F_n S_{21} & -F_n S_{22} \\ F_n^T S_{11} & F_n^T S_{12} \end{bmatrix}. \tag{50}$$

$B := [B_{ij}]_{i,j=1}^2$ is reduced by adding

$$\Delta B := \begin{bmatrix} \Delta B_{11} & \Delta B_{12} \\ \Delta B_{21} & \Delta B_{22} \end{bmatrix} = \begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & G_n \\ -G_n^T & 0 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

$$= \begin{bmatrix} -R_{21}^T G_m^T + G_n S_{21} & R_{11}^T G_m + G_n S_{22} \\ -R_{22}^T G_m^T - G_n^T S_{11} & R_{12}^T G_m - G_n^T S_{12} \end{bmatrix},$$

where S_{ij} and R_{ij} , $i,j = 1,2$ satisfy (50).

We reduce each pair of blocks independently. First we reduce B_{11}. Using the equality $R_{21}^T F_m^T = F_n S_{21}$ we obtain that

$$S_{21} = \begin{bmatrix} & & Q & \\ a_1 & \cdots & & a_m \end{bmatrix}, R_{21}^T = \begin{bmatrix} & & b_1 \\ Q & & \vdots \\ & & b_n \end{bmatrix}, \text{where } Q = [q_{ij}] \text{ is any } n\text{-by-}m \text{ matrix.}$$

Therefore

$$\Delta B_{11} = -R_{21}^T G_m^T + G_n S_{21} = -\begin{bmatrix} & & b_1 \\ Q & & \vdots \\ & & b_n \end{bmatrix} G_m^T + G_n \begin{bmatrix} & & Q & \\ a_1 & \cdots & & a_m \end{bmatrix}$$

$$= \begin{bmatrix} q_{21} - q_{12} & q_{22} - q_{13} & \cdots & q_{2,m-1} - q_{1m} & q_{2m} - b_1 \\ q_{31} - q_{22} & q_{32} - q_{23} & \cdots & q_{3,m-1} - q_{2m} & q_{3m} - b_2 \\ q_{41} - q_{32} & q_{42} - q_{33} & \cdots & q_{4,m-1} - q_{3m} & q_{4m} - b_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ q_{n1} - q_{n-1,2} & q_{n2} - q_{n-1,3} & \cdots & q_{n,m-1} - q_{n-1,m} & q_{nm} - b_{m-1} \\ a_1 - q_{n2} & a_2 - q_{n3} & \cdots & a_{n-1} - q_{nm} & a_n - b_m \end{bmatrix}. \quad (51)$$

We can set each anti-diagonal of B_{11} to zero independently by adding the corresponding anti-diagonal of ΔB_{11}. Thus we can reduce B_{11} by adding ΔB_{11} to zero.

Now to the pair (A_{12}, B_{12}): To preserve A_{12}, we take R_{11} and S_{22} such that $R_{11}^T F_m = -F_n S_{22}$ thus

$$S_{22} = \begin{bmatrix} & & & 0 \\ & -R_{11}^T & & \vdots \\ & & & 0 \\ b_1 & \cdots & b_m & b_{m+1} \end{bmatrix},$$

where R_{11}^T is any n-by-m matrix. Thus

$$\Delta B_{12} = R_{11}^T G_m + G_n S_{22} = R_{11}^T G_m + G_n \begin{bmatrix} & & & 0 \\ & -R_{11}^T & & \vdots \\ & & & 0 \\ b_1 & \dots & b_m & b_{m+1} \end{bmatrix}$$

$$= \begin{bmatrix} -r_{21} & r_{11} - r_{22} & r_{12} - r_{23} & \dots & r_{1,m-1} - r_{2m} & r_{1m} \\ -r_{31} & r_{21} - r_{32} & r_{22} - r_{33} & \dots & r_{2,m-1} - r_{3m} & r_{2m} \\ -r_{41} & r_{31} - r_{42} & r_{32} - r_{43} & \dots & r_{3,m-1} - r_{4m} & r_{3m} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -r_{n1} & r_{n-1,1} - r_{n2} & r_{n-1,2} - r_{n3} & \dots & r_{n-1,m-1} - r_{nm} & r_{n-1m} \\ b_1 & r_{n1} + b_2 & r_{n2} + b_3 & \dots & r_{n,m-1} + b_m & r_{nm} + b_{m+1} \end{bmatrix}.$$

If $m + 1 \geq n$ then we can set B_{12} to zero by adding ΔB_{12}. If $n > m + 1$ then we cannot set B_{12} to zero. Then we reduce it diagonal-wise starting from the top-right corner. By adding the first m and the last $m + 1$ diagonals of ΔB_{12} we set the corresponding diagonals of B_{12} to zeros. We can set the remaining $n - m - 1$ diagonals of B_{12} to zeros, except the last element of each of them. Hence (A_{12}, B_{12}) is reduced to $(0, 0_{m+1,n}^{\boxminus T})$ by adding $\Delta(A_{12}, B_{12})$.

(A_{21}, B_{21}) is reduced in the same way (up to the transposition) as (A_{12}, B_{12}). Hence it can be reduced to the form $(0, 0_{n+1,m}^{\boxminus})$.

Consider (A_{22}, B_{22}). We reduce A_{22} to the form 0_* by adding $\Delta A_{22} = R_{12}^T F_m - F_n^T S_{12}$. To preserve A_{22}, we must hereafter take R_{12} and S_{12} such that $R_{12}^T F_m = F_n^T S_{12}$ thus

$$R_{12}^T = \begin{bmatrix} & Q & \\ 0 & \dots & 0 \end{bmatrix}, S_{12} = \begin{bmatrix} Q & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{bmatrix}, \text{ where } Q = [q_{ij}] \text{ is any } n\text{-by-}m \text{ matrix.}$$

Therefore,

$$\Delta B_{22} = R_{12}^T G_m - G_n^T S_{12} = \begin{bmatrix} & Q & \\ 0 & \dots & 0 \end{bmatrix} G_m - G_n^T \begin{bmatrix} Q & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & q_{11} & q_{12} & \dots & q_{1,m-1} & q_{1m} \\ -q_{11} & q_{21} - q_{12} & q_{22} - q_{13} & \dots & q_{2,m-1} - q_{1m} & q_{2m} \\ -q_{21} & q_{31} - q_{22} & q_{32} - q_{23} & \dots & q_{3,m-1} - q_{2m} & q_{3m} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -q_{n-1,1} & q_{n1} - q_{n-1,2} & q_{n2} - q_{n-1,3} & \dots & q_{n,m-1} - q_{n-1,m} & q_{nm} \\ -q_{n1} & -q_{n2} & -q_{n3} & \dots & -q_{nm} & 0 \end{bmatrix}.$$

By adding ΔB_{22}, we can set each element of B_{22} to zero except the elements in the first column and the last row (or, alternatively, the elements in the first row and the last column).

Summing up the results, we have that $\mathcal{D}(\mathcal{L}_n, \mathcal{L}_m)$ is of the form (19).

3.4. Off-diagonal blocks of $\mathcal{D}$ that correspond to summands of $(A,B)_{can}$ of different types

Finally, we verify the condition (ii) of Lemma 3.3 for off-diagonal blocks of $\mathcal{D}$ defined in Theorem 2.1(iii); the diagonal blocks of their horizontal and vertical strips contain summands of $(A,B)_{\text{can}}$ of different types.

3.4.1. Pairs of blocks $\mathcal{D}(\mathcal{H}_n(\lambda),\mathcal{K}_m)$

Due to Lemma 3.3(ii), it suffices to prove that each group of four matrices $((A,B),(-A^T,-B^T))$ can be reduced to exactly one group of the form (20) by adding

$$(R^T\mathcal{K}_m + \mathcal{H}_n(\lambda)S, S^T\mathcal{H}_n(\lambda) + \mathcal{K}_m R), \quad R \in \mathbb{C}^{2m \times 2n}, \ S \in \mathbb{C}^{2n \times 2m}.$$

Obviously, if we reduce (A,B) then the second pair will be reduced automatically. We have

$$\Delta(A,B) = R^T\mathcal{K}_m + H_m(\lambda)S$$
$$= \left(R^T\begin{bmatrix} 0 & J_m(0) \\ -J_m(0)^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}S, R^T\begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix}S\right).$$

It is clear that we can set A to zero. To preserve A, we must hereafter take $R = [R_{ij}]_{i,j=1}^2$ and $S = [S_{ij}]_{i,j=1}^2$ such that

$$R^T\begin{bmatrix} 0 & J_m(0) \\ -J_m(0)^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}S = 0,$$

or equivalently

$$S = \begin{bmatrix} -R_{22}^T J_m(0)^T & R_{12}^T J_m(0) \\ R_{21}^T J_m(0)^T & -R_{11}^T J_m(0) \end{bmatrix}.$$

Therefore $B = [B_{ij}]_{i,j=1}^2$ is reduced by adding

$$\Delta B = \begin{bmatrix} \Delta B_{11} & \Delta B_{12} \\ \Delta B_{21} & \Delta B_{22} \end{bmatrix}$$
$$= \begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix}\begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix}\begin{bmatrix} -R_{22}^T J_m(0)^T & R_{12}^T J_m(0) \\ R_{21}^T J_m(0)^T & -R_{11}^T J_m(0) \end{bmatrix}$$
$$= \begin{bmatrix} -R_{21}^T + J_n(\lambda)R_{21}^T J_m(0)^T & R_{11}^T - J_n(\lambda)R_{11}^T J_m(0) \\ -R_{22}^T + J_n(\lambda)^T R_{22}^T J_m(0)^T & R_{12}^T - J_n(\lambda)^T R_{12}^T J_m(0) \end{bmatrix}.$$

The block B_{11} is reduced to zero by adding

$$\Delta B_{11} = -R_{21}^T + J_n(\lambda)R_{21}^T J_m(0)^T$$
$$= \begin{cases} -r_{ij} + \lambda r_{i,j+1} + r_{i+1,j+1} & \text{if } 1 \le i \le n-1, 1 \le j \le m-1, \\ -r_{ij} + \lambda r_{i,j+1} & \text{if } 1 \le j \le m-1, i = n, \\ -r_{ij} & \text{if } 1 \le i \le n, j = m, \end{cases}$$

because it results in a square system of nm equations that has a solution.

The reduction of the other blocks follows from above since

$$R_{11}^T - J_n(\lambda) R_{11}^T J_m(0) = -R_{21}^T Z + J_n(\lambda) R_{21}^T Z Z J_m(0)^T Z,$$

$$-R_{22}^T + J_n(\lambda)^T R_{22}^T J_m(0)^T = -Z R_{21}^T Z + Z J_n(\lambda) Z Z R_{21}^T Z Z J_m(0)^T Z,$$

$$R_{12}^T - J_n(\lambda)^T R_{12}^T J_m(0) = -Z R_{21}^T + Z J_n(\lambda) Z Z R_{21}^T J_m(0)^T,$$

where the matrices Z (see (44)) are of the corresponding sizes.

Altogether, we have that $\mathcal{D}(\mathcal{H}_n(\lambda), \mathcal{K}_m)$ is zero.

3.4.2. Pairs of blocks $\mathcal{D}(\mathcal{H}_n(\lambda), \mathcal{L}_m)$

Due to Lemma 3.3(ii), it suffices to prove that each group of four matrices $((A, B), (-A^T, -B^T))$ can be reduced to the group of the form (21) by adding

$$(R^T \mathcal{L}_m + \mathcal{H}_n(\lambda) S, S^T \mathcal{H}_n(\lambda) + \mathcal{L}_m R), \quad S \in \mathbb{C}^{2n \times 2m+1}, \ R \in \mathbb{C}^{2m+1 \times 2n}.$$

Obviously, if we only reduce (A, B), then $(-A^T, -B^T)$ will be reduced automatically. We have

$$\Delta(A, B) = R^T \mathcal{L}_m + \mathcal{H}_n(\lambda) S$$

$$= \left(R^T \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S, R^T \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} S \right).$$

It is easy to check that we can set A to zero. To preserve A, we must hereafter take $R = [R_{ij}]_{i,j=1}^2$ and $S = [S_{ij}]_{i,j=1}^2$ such that

$$R^T \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S = 0, \text{ or equivalently } S = \begin{bmatrix} -R_{22}^T F_m^T & R_{12}^T F_m \\ R_{21}^T F_m^T & -R_{11}^T F_m \end{bmatrix}.$$

Thus $B = [B_{ij}]_{i,j=1}^2$ is reduced by adding

$$\Delta B = \begin{bmatrix} \Delta B_{11} & \Delta B_{12} \\ \Delta B_{21} & \Delta B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} \begin{bmatrix} -R_{22}^T F_m^T & R_{12}^T F_m \\ R_{21}^T F_m^T & -R_{11}^T F_m \end{bmatrix}$$

$$= \begin{bmatrix} -R_{21}^T G_m^T + J_n(\lambda) R_{21}^T F_m^T & R_{11}^T G_m - J_n(\lambda) R_{11}^T F_m \\ -R_{22}^T G_m^T + J_n(\lambda)^T R_{22}^T F_m^T & R_{12}^T G_m - J_n(\lambda)^T R_{12}^T F_m \end{bmatrix}.$$

First, adding

$$\Delta B_{11} = -R_{21}^T G_m^T + J_n(\lambda) R_{21}^T F_m^T =$$

$$\begin{bmatrix} -r_{12} + \lambda r_{11} + r_{21} & -r_{13} + \lambda r_{12} + r_{22} & \dots & -r_{1,m+1} + \lambda r_{1m} + r_{2m} \\ -r_{22} + \lambda r_{21} + r_{31} & -r_{23} + \lambda r_{22} + r_{32} & \dots & -r_{2,m+1} + \lambda r_{2m} + r_{3m} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -r_{n-1,2} + \lambda r_{n-1,1} + r_{n1} & -r_{n-1,3} + \lambda r_{n-1,2} + r_{n2} & \dots & -r_{n-1,m+1} + \lambda r_{n-1,m} + r_{nm} \\ -r_{n2} + \lambda r_{n1} & -r_{n3} + \lambda r_{n2} & \dots & -r_{n,m+1} + \lambda r_{nm} \end{bmatrix},$$

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we can set B_{11} to zero as follows. For the last (n-th) row of B_{11} we have the following system of equations

$$
\begin{bmatrix}
\lambda & -1 & & & \\
 & \lambda & -1 & & \\
 & & \ddots & \ddots & \\
 & & & \lambda & -1
\end{bmatrix}
\begin{bmatrix}
r_{n1} \\ r_{n2} \\ \vdots \\ r_{nm} \\ r_{n,m+1}
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ \vdots \\ b_m
\end{bmatrix}
\tag{52}
$$

which has a solution. For the $(n-1)$-th row we have

$$
\begin{bmatrix}
\lambda & -1 & & & \\
 & \lambda & -1 & & \\
 & & \ddots & \ddots & \\
 & & & \lambda & -1
\end{bmatrix}
\begin{bmatrix}
r_{n-1,1} \\ r_{n-1,2} \\ \vdots \\ r_{n-1,m} \\ r_{n-1,m+1}
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ \vdots \\ b_m
\end{bmatrix}
-
\begin{bmatrix}
r_{n1} \\ r_{n2} \\ \vdots \\ r_{nm}
\end{bmatrix}.
\tag{53}
$$

The variables $r_{n1}, r_{n2}, \ldots, r_{nm}$ are known from (52), thus (53) becomes a system of the type (52) and the system (53) has a solution. Repeating this reduction to every row from the bottom to the top, we set B_{11} to zero.

The block B_{21} is reduced like the block B_{11} and thus we omit this verification.

Now we turn to the reduction of B_{12} and B_{22}. It suffices to consider only B_{12}. We have

$$
\Delta B_{12} = R_{11}^T G_m - J_n(\lambda) R_{11}^T F_m
$$
$$
=
\begin{bmatrix}
-\lambda r_{11} - r_{21} & r_{11} - \lambda r_{12} - r_{22} & \ldots & r_{1,m-1} - \lambda r_{1m} - r_{2m} & r_{1m} \\
-\lambda r_{21} - r_{31} & r_{21} - \lambda r_{22} - r_{32} & \ldots & r_{2,m-1} - \lambda r_{2m} - r_{3m} & r_{2m} \\
\hdotsfor{5} \\
-\lambda r_{n-1,1} - r_{n1} & r_{n-1,1} - \lambda r_{n-1,2} - r_{n2} & \ldots & r_{n-1,m-1} - \lambda r_{n-1,m} - r_{nm} & r_{n-1,m} \\
-\lambda r_{n1} & r_{n1} - \lambda r_{n2} & \ldots & r_{n,m-1} - \lambda r_{nm} & r_{nm}
\end{bmatrix}.
$$

Adding ΔB_{12} we reduce B_{12} to the form $0^{\leftarrow}$.

Summing up the results for all the blocks, we have that $\mathcal{D}(\mathcal{H}_n(\lambda), \mathcal{L}_m)$ is equal to (21).

3.4.3. Pairs of blocks $\mathcal{D}(\mathcal{K}_n, \mathcal{L}_m)$

Due to Lemma 3.3(ii), it suffices to prove that each group of four matrices $((A, B), (-A^T, -B^T))$ can be reduced to the group of the form (22) by adding

$$
(R^T \mathcal{L}_m + \mathcal{K}_n S, S^T \mathcal{K}_n + \mathcal{L}_m R), \quad S \in \mathbb{C}^{2n \times 2m+1}, \ R \in \mathbb{C}^{2m+1 \times 2n}.
$$

As in the previous sections, we reduce only (A, B) and $(-A^T, -B^T)$ is reduced automatically. We have

$$\Delta(A, B) = R^T \mathcal{L}_m + \mathcal{K}_n S$$

$$= \left(R^T \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(0) \\ -J_n(0)^T & 0 \end{bmatrix} S, \; R^T \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S \right).$$

It is clear that we can set B to zero. To preserve B, we must hereafter take $R = [R_{ij}]_{i,j=1}^2$ and $S = [S_{ij}]_{i,j=1}^2$ such that

$$R^T \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S = 0, \text{ or equivalently } S = \begin{bmatrix} -R_{22}^T G_m^T & R_{12}^T G_m \\ R_{21}^T G_m^T & -R_{11}^T G_m \end{bmatrix}.$$

Hence $A = [A_{ij}]_{i,j=1}^2$ is reduced by adding

$$\Delta A = \begin{bmatrix} \Delta A_{11} & \Delta A_{12} \\ \Delta A_{21} & \Delta A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(0) \\ -J_n(0)^T & 0 \end{bmatrix} \begin{bmatrix} -R_{22}^T G_m^T & R_{12}^T G_m \\ R_{21}^T G_m^T & -R_{11}^T G_m \end{bmatrix}$$

$$= \begin{bmatrix} -R_{21}^T F_m^T + J_n(0) R_{21}^T G_m^T & R_{11}^T F_m - J_n(0) R_{11}^T G_m \\ -R_{22}^T F_m^T + J_n(0)^T R_{22}^T G_m^T & R_{12}^T F_m - J_n(0)^T R_{12}^T G_m \end{bmatrix}.$$

First we reduce the block A_{11} (A_{21} is reduced in the same way). We have

$$\Delta A_{11} = -R_{21}^T F_m^T + J_n(0) R_{21}^T G_m^T$$

$$= \begin{bmatrix} -r_{11} + r_{22} & -r_{12} + r_{23} & -r_{13} + r_{24} & \dots & -r_{1m} + r_{2,m+1} \\ -r_{21} + r_{32} & -r_{22} + r_{33} & -r_{23} + r_{34} & \dots & -r_{2m} + r_{3,m+1} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -r_{n-1,1} + r_{n2} & -r_{n-1,2} + r_{n3} & -r_{n-1,3} + r_{n4} & \dots & -r_{n-1,m} + r_{n,m+1} \\ -r_{n1} & -r_{n2} & -r_{n4} & \dots & -r_{nm} \end{bmatrix},$$

and thus we reduce each diagonal of A_{11} independently. For each of the first m diagonals, starting from the bottom-left corner, we have a system of type (43) which has a solution, and for the remaining diagonals we have the system of type (42) which has a solution too. Thus adding ΔA_{11} we set A_{11} to zero.

Last, we reduce the blocks A_{12} and A_{22} and it is enough to consider only A_{12}. We have

$$\Delta A_{12} = R_{11}^T F_m - J_n(0) R_{11}^T G_m$$

$$= \begin{bmatrix} r_{11} & r_{12} - r_{21} & r_{13} - r_{22} & \dots & r_{1m} - r_{2,m-1} & -r_{2m} \\ r_{21} & r_{22} - r_{31} & r_{13} - r_{32} & \dots & r_{2m} - r_{3,m-1} & -r_{2m} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ r_{n-1,1} & r_{n-1,2} - r_{n1} & r_{n-1,3} - r_{n2} & \dots & r_{n-1,m} - r_{n,m-1} & -r_{nm} \\ r_{n1} & r_{n2} & r_{n3} & \dots & r_{nm} & 0 \end{bmatrix}.$$

Adding ΔA_{12} we reduce A_{12} to the form $0^{\rightarrow}$.

Summing up the results for all blocks we have that $\mathcal{D}(\mathcal{K}_n, \mathcal{L}_m)$ is equal to (22).

Acknowledgements

References

[1] V.I. Arnold, On matrices depending on parameters, Russian Math. Surveys, 26 (2) (1971) 29–43.

[2] V.I. Arnold, Lectures on bifurcations in versal families, Russian Math. Surveys, 27 (5) (1972) 54–123.

[3] V.I. Arnold, Geometrical methods in the theory of ordinary differential equations, Springer-Verlag, New York, 1988.

[4] L. Batzke, Generic low rank perturbations of structured regular matrix pencils and structured matrices, Doctoral Thesis, TU Berlin, 2015.

[5] R. Byers, V. Mehrmann, and H. Xu, A structured staircase algorithm for skew-symmetric/symmetric pencils, Electron. Trans. Numer. Anal. 26 (2007) 1–33.

[6] T.J. Bridges, S. Reich, Multi-symplectic integrators: numerical schemes for Hamiltonian PDEs that conserve symplecticity, Physics Letters A, 284(4–5), (2001) 184–193.

[7] T. Brüll and V. Mehrmann, STCSSP: A FORTRAN 77 routine to compute a structured staircase form for a (skew-)symmetric/(skew-)symmetric matrix pencil, Preprint 31, Instituts für Mathematik Technische Universitält, 2007.

[8] F. De Terán and F.M. Dopico, The solution of the equation $XA + AX^T = 0$ and its application to the theory of orbits, Linear Algebra Appl. 434 (2011) 44–67.

[9] F. De Terán and F.M. Dopico, The equation $XA + AX^* = 0$ and the dimension of *congruence orbits, Electr. J. Linear Algebra, 22 (2011) 448–465.

[10] A.R. Dmytryshyn, Miniversal Deformations of Pairs of Skew-symmetric Forms, Master Thesis, Kiev National University, Kiev, 2010.

[11] A.R. Dmytryshyn, Miniversal deformations of pairs of symmetric forms, Manuscript, 2011, arXiv:1104.2530.

[12] A. Dmytryshyn, S. Johansson, and B. Kågström, Codimension computations of congruence orbits of matrices, skew-symmetric and symmetric matrix pencils using Matlab, Technical report UMINF 13.18, Dept. of Computing Science, Umeå University, Sweden, 2013.

[13] A. Dmytryshyn, V. Futorny, B. Kågström, L. Klimenko, and V.V. Sergeichuk, Change of the congruence canonical form of 2-by-2 and 3-by-3 matrices under perturbations and bundles of matrices under congruence, Linear Algebra Appl., 469 (2015) 305–334.

[14] A.R. Dmytryshyn, V. Futorny, and V.V. Sergeichuk, Miniversal deformations of matrices of bilinear forms, Linear Algebra Appl., 436(7) (2012) 2670–2700.

[15] A. Dmytryshyn, V. Futorny, and V.V. Sergeichuk, Miniversal deformations of matrices under *congruence and reducing transformations, Linear Algebra Appl., 446 (2014) 388–420.

[16] A. Dmytryshyn and B. Kågström, Orbit closure hierarchies of skew-symmetric matrix pencils, SIAM J. Matrix Anal. Appl., 35(4) (2014) 1429–1443.

[17] A. Dmytryshyn, B. Kågström, and V.V. Sergeichuk, Skew-symmetric matrix pencils: Codimension counts and the solution of a pair of matrix equations, Linear Algebra Appl., 438(8) (2013) 3375–3396.

[18] A. Dmytryshyn, B. Kågström, and V.V. Sergeichuk, Symmetric matrix pencils: Codimension counts and the solution of a pair of matrix equations, Electron. J. Linear Algebra, 27 (2014) 1–18.

[19] A. Edelman, E. Elmroth, and B. Kågström, A geometric approach to perturbation theory of matrices and matrix pencils. Part I: Versal deformations, SIAM J. Matrix Anal. Appl., 18(3) (1997) 653–692.

[20] A. Edelman, E. Elmroth, and B. Kågström, A geometric approach to perturbation theory of matrices and matrix pencils. Part II: A stratification-enhanced staircase algorithm, SIAM J. Matrix Anal. Appl., 20 (1999) 667–669.

[21] E. Elmroth, S. Johansson, and B. Kågström, Stratification of controllability and observability pairs theory and use in applications, SIAM J. Matrix Anal. Appl., 31(2) (2009) 203–226.

[22] V. Futorny, L. Klimenko, and V.V. Sergeichuk, Change of the *congruence canonical form of 2-by-2 matrices under perturbations, Electr. J. Linear Algebra 27 (2014) 146–154.

[23] M.I. Garcia-Planas and V.V. Sergeichuk, Simplest miniversal deformations of matrices, matrix pencils, and contragredient matrix pencils, Linear Algebra Appl., 302–303 (1999) 45–61.

[24] R.A. Horn and C.R. Johnson, Matrix Analysis, Cambridge U. P., Cambridge, 1985.

[25] P. Johansson, Matrix canonical structure toolbox, Technical report UMINF 06.15, Dept. of Computing Science, Umeå University, Sweden, 2006.

[26] A.G. Kurosh, Higher algebra, Mir Publishers, 1972.

[27] B. Kågström, S. Johansson, and P. Johansson, StratiGraph Tool: Matrix Stratification in Control Applications. In L. Biegler, S. L. Campbell, and V. Mehrmann, editors, Control and Optimization with Differential-Algebraic Constraints, chapter 5. SIAM Publications, 2012.

[28] P.Y. Li, R. Horowitz, Passive Velocity Field Control of Mechanical Manipulators, IEEE Transactions on Robotics and Automation, 15 (4) (1999) 751 –763.

[29] A. Gohberg, P. Lancaster, and L. Rodman, Invariant Subspaces of Matrices with Applications, Vol. 51. SIAM, 1986.

[30] A.A. Mailybaev, Transformation of families of matrices to normal forms and its application to stability theory, SIAM J. Matrix Anal. Appl., 21 (2000) 396–417.

[31] A.I. Mal'cev, Foundations of linear algebra. Translated from the Russian by Thomas Craig Brown, J. B. Roberts, ed., W. H. Freeman & Co., San Francisco, 1963.

[32] D.S. Mackey, N. Mackey, and F. Tisseur, Polynomial Eigenvalue Problems: Theory, Computation, and Structure, in P. Benner et. al., Numerical Algebra, Matrix Theory, Differential-Algebraic Equations and Control Theory, chapter 12, Springer, 2015.

[33] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann, Skew-symmetric matrix polynomials and their Smith forms, Linear Algebra Appl., 438(12) (2013) 4625–4653.

[34] P.J. Olver, Canonical forms for compatible biHamiltonian systems. In I. Antoniou and F. Lambert, editors, Solitons and Chaos, pp. 171–179, Springer-Verlag, New York, 1991.

[35] L. Rodman, Comparison of congruences and strict equivalences for real, complex, and quaternionic matrix pencils with symmetries,. Electron. J. Linear Algebra, 16 (2007), 248–283.

[36] R. Scharlau, Paare alternierender Formen, Math. Z. 147 (1976) 13–19.

[37] R.C. Thompson, Pencils of complex and real symmetric and skew matrices, Linear Algebra Appl., 147 (1991) 323–371.