

Structure preserving stratification of skew-symmetric matrix polynomials

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Abstract

We study how elementary divisors and minimal indices of a skewsymmetric matrix polynomial of odd degree may change under small perturbations of the matrix coefficients. We investigate these changes qualitatively by constructing the stratifications (closure hierarchy graphs) of orbits and bundles for skew-symmetric linearizations. We also derive the necessary and sufficient conditions for the existence of a skew-symmetric matrix polynomial with prescribed degree, elementary divisors, and minimal indices.

1 Introduction

Applications of matrix polynomials [28, 29, 34, 40, 43] stimulates rapid developments of the corresponding theories [8, 9, 10, 31, 38], computational techniques [32, 35, 40], and software [5, 33]. In a number of cases, elementary divisors and minimal indices, i.e., the canonical structure information, of matrix polynomials provide a complete understanding of the properties and behaviours of the underlying physical systems and thus are the actual objects of interest. This canonical structure information is sensitive to perturbations of the matrix-coefficients of the polynomials, e.g., the eigenvalues may coalesce or split apart, appear or disappear. In general, these problems are called ill-posed, meaning that small perturbations in the input parameters may lead to big changes in the answers. One way to study qualitatively

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how small perturbations can change the canonical structure information of matrix polynomials is to construct the stratifications, i.e., closure hierarchy graphs, of the corresponding orbits or bundles. Each node of such a graph represents matrix polynomials with a certain canonical structure information and there is an edge from one node to another if we can perturb a polynomial associated with the first node such that its canonical structure information becomes equal to a polynomial associated with the second node. The ways to construct such graphs are already known for various matrix problems: matrices under similarity (i.e., Jordan canonical form) [11, 24, 39], matrix pencils (i.e., Kronecker canonical form) [24], skew-symmetric matrix pencils [20], controllability and observability pairs [25], as well as state-space system pencils [18]. Many of these results are already implemented in Stratigraph [30] which is a java-based tool developed to construct and visualize such closure hierarchy graphs. The Matrix Canonical Structure (MCS) Toolbox for Matlab [17, 30, 42] was also developed for simplifying the work with the matrices in their canonical forms and connecting Matlab with StratiGraph. For more details on each of these cases we recommend the corresponding papers and their references; some applications in control theory are described in [33].

The paper [31] is the first one to investigate the possible changes of the canonical structure information for matrix polynomials, in particular, the authors construct the stratifications for the first or second companion linearizations of full rank polynomials. These results from [31] have been extended to matrix polynomials of any ranks in [19]. Notably linearizations are typically used for computing the canonical structure information of matrix polynomials.

Sometimes, given matrix polynomials have additional structures that may be explored in computations, e.g., they are (skew-)symmetric, (skew-)Hermitian, palindromic, alternating, etc. Therefore of particular interest are structure preserving linearizations [1, 34, 36, 37], solutions of structured eigenvalue problems [32], and structured canonical forms [6, 7, 22, 41].

In this paper, we study how elementary divisors and minimal indices of skew-symmetric matrix polynomials of odd degrees may change under small perturbations, by constructing the orbit and bundle stratifications of their skew-symmetric linearizations. This requires a number of other results including the necessary and sufficient conditions for a skew-symmetric matrix polynomial with certain degree and canonical structure information to exist, see Theorem 5 which is based on [10, 31]; the skew-symmetric strong linearization templates [37] and how the minimal indices of such linearizations are related to the minimal indices of the polynomials [8]; the relation between perturbations of the linearizations and perturbations of matrix polynomials, Theorem 8, see also [31]; the stratifications of skew-symmetric matrix pencils [20] and computations of their codimensions [12, 21].

Let us extend on the last paragraph and sketch a possible scheme for solving the stratification problems for (structured) linearizations of matrix polynomials. To investigate how the elementary divisors and minimal indices of a (structured) matrix polynomial change under small perturbations we need

- to know necessary and sufficient conditions for a (structured) matrix polynomial with certain degree and canonical structure information to exist;
- to have a (structured) strong linearization of the matrix polynomials;
- to know how the minimal indices of the (structured) matrix polynomials and linearizations are related;
- to prove that there is a correspondence between perturbations of the (structured) matrix polynomials and their linearizations;
- to be able to stratify the matrix pencils (with the corresponding structure).

The last two bullets in the list can be combined under the more general title "to stratify the (structured) linearizations" and we rather specify one possible way to do it (though so far it is the only used/known way). Hopefully this scheme will provide possibilities including the identification of "gaps" for solving the stratification problem for other types of matrix polynomials.

All matrices that we consider are over the field of complex numbers.

2 Skew-symmetric matrix pencils

First, we recall the canonical form of skew-symmetric matrix pencils under congruence. We follow the notations and style of [20].

For each $k = 1, 2, \ldots$, define the $k \times k$ matrices

$$J_{k}(\mu) := \begin{bmatrix} \mu & 1 & & \\ & \mu & \ddots & \\ & & \ddots & 1 \\ & & & \mu \end{bmatrix}, \qquad I_{k} := \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix},$$

where $\mu \in \mathbb{C}$, and for each k = 0, 1, ..., define the $k \times (k + 1)$ matrices

$$F_k := \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}, \qquad G_k := \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}.$$

All non-specified entries of $J_k(\mu)$, I_k , F_k , and G_k are zeros.

An $n \times n$ matrix pencil $A - \lambda B$ with $A = -A^T$ and $B = -B^T$ is called skew-symmetric. A skew-symmetric matrix pencil $A - \lambda B$ is congruent to $C - \lambda D$ if and only if there is a nonsingular matrix S such that $S^T A S = C$ and $S^T B S = D$. Recall that congruence preserves skew symmetry. The set of matrix pencils congruent to a skew-symmetric matrix pencil $A - \lambda B$ forms a manifold in the complex $n^2 - n$ dimensional space (A has n(n-1)/2)independent parameters and so does B). This manifold is the *orbit* of $A - \lambda B$ under the action of the group $GL_n(\mathbb{C})$ on the space of skew-symmetric matrix pencils by congruence:

$$O_{A-\lambda B}^{c} = \{ S^{T}(A - \lambda B) S : S \in GL_{n}(\mathbb{C}) \}.$$
(1)

The dimension of $O_{A-\lambda B}^{c}$ is the dimension of its tangent space

$$T^{c}_{A-\lambda B} \coloneqq \{ (X^{T}A + AX) - \lambda (X^{T}B + BX) : X \in \mathbb{C}^{n \times n} \}$$
(2)

at the point $A - \lambda B$. The orthogonal complement (in the space of all skewsymmetric matrix pencils) to $T^{c}_{A-\lambda B}$, with respect to the Frobenius inner product

$$\langle A - \lambda B, C - \lambda D \rangle = \operatorname{trace}(AC^* + BD^*),$$
 (3)

is called the normal space to this orbit. The dimension of the normal space is the *codimension* of the congruence orbit of $A - \lambda B$, denoted $\operatorname{cod} O_{A-\lambda B}^c$, and is equal to $n^2 - n$ minus the dimension of the congruence orbit of $A - \lambda B$. Recently, the explicit expressions for the codimensions of congruence orbits of skew-symmetric matrix pencils were derived in [21]. **Theorem 1.** [41] Each skew-symmetric $n \times n$ matrix pencil $A - \lambda B$ is congruent to a direct sum, determined uniquely up to permutation of summands, of pencils of the form

$$H_{h}(\mu) \coloneqq \begin{bmatrix} 0 & J_{h}(\mu) \\ -J_{h}(\mu)^{T} & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & I_{h} \\ -I_{h} & 0 \end{bmatrix}, \quad \mu \in \mathbb{C},$$
$$K_{k} \coloneqq \begin{bmatrix} 0 & I_{k} \\ -I_{k} & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & J_{k}(0) \\ -J_{k}(0)^{T} & 0 \end{bmatrix},$$
$$M_{m} \coloneqq \begin{bmatrix} 0 & F_{m} \\ -F_{m}^{T} & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & G_{m} \\ -G_{m}^{T} & 0 \end{bmatrix}.$$

Therefore every skew-symmetric pencil $A - \lambda B$ is congruent to one in the following direct sum form

$$A - \lambda B = \bigoplus_{j} \bigoplus_{i} H_{h_i}(\mu_j) \oplus \bigoplus_{i} K_{k_i} \oplus \bigoplus_{i} M_{m_i}, \qquad (4)$$

where the first direct (double) sum corresponds to all distinct eigenvalues $\mu_j \in \mathbb{C}$. The blocks $H_k(\mu)$ and K_k correspond to the finite and infinite eigenvalues, respectively, and altogether form the regular part of $A - \lambda B$. The blocks M_k correspond to pairs of the column and row minimal indices, and form the singular part of the matrix pencil.

3 Skew-symmetric matrix polynomials with prescribed invariants

We consider skew-symmetric $n \times n$ matrix polynomials $P(\lambda)$ of degree d over \mathbb{C} , i.e.,

$$P(\lambda) = \lambda^d A_d + \dots + \lambda A_1 + A_0, \quad A_d \neq 0, \ A_i^T = -A_i, \ A_i \in \mathbb{C}^{n \times n} \text{ for } i = 0, \dots, d.$$

Two matrix polynomials $P(\lambda)$ and $Q(\lambda)$ are called *unimodularly con*gruent if and only if there exists a unimodular matrix polynomial $F(\lambda)$ (i.e., det $F(\lambda) \in \mathbb{C} \setminus \{0\}$) such that $F(\lambda)^T P(\lambda) F(\lambda) = R(\lambda)$, see more details in [37]. In the following theorem we recall the canonical form for skew-symmetric matrix polynomials under unimodular congruence, derived in [37]. **Theorem 2.** [37] Let $P(\lambda)$ be a skew-symmetric $n \times n$ matrix polynomial. Then there exists $r \in \mathbb{N}$ with $2r \leq n$ and a unimodular matrix polynomial $F(\lambda)$ such that

$$F(\lambda)^T P(\lambda) F(\lambda) = \begin{bmatrix} 0 & g_1(\lambda) \\ -g_1(\lambda) & 0 \end{bmatrix} \oplus \dots \oplus \begin{bmatrix} 0 & g_r(\lambda) \\ -g_r(\lambda) & 0 \end{bmatrix} \oplus 0_{n-2r} =: S(\lambda),$$

where g_j is monic for j = 1, ..., r and $g_j(\lambda)$ divides $g_{j+1}(\lambda)$ for j = 1, ..., r-1. Moreover, the canonical form $S(\lambda)$ is unique.

Recall also that two matrix polynomials $P(\lambda)$ and $Q(\lambda)$ are called *unimodularly equivalent* if and only if there exist unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ (i.e., $\det U(\lambda), \det V(\lambda) \in \mathbb{C} \setminus \{0\}$) such that $U(\lambda)P(\lambda)V(\lambda) = Q(\lambda)$. Every matrix polynomial is unimodulary equivalent to its Smith form [26, 37] and every skew-symmetric matrix polynomial is unimodularly congruent to the canonical form in Theorem 2 which in fact is the skew-symmetrically structured Smith form. In particular, this means that the invariants for skew-symmetric matrix polynomials under unimodular congruence are the same as under unimodular equivalence, see also [37].

Every $g_j(\lambda)$ in $S(\lambda)$ from Theorem 2 can be uniquely factored as

$$g_j(\lambda) = (\lambda - \alpha_1)^{\delta_{j1}} \cdot (\lambda - \alpha_2)^{\delta_{j2}} \cdot \ldots \cdot (\lambda - \alpha_{l_j})^{\delta_{jl_j}},$$

where the integers $l_j \ge 0$, and $\delta_{j1}, \ldots, \delta_{jl_j} > 0$. If $l_j = 0$ then $g_j(\lambda) = 1$. The numbers $\alpha_1, \ldots, \alpha_{l_j} \in \mathbb{C}$ are the finite eigenvalues of $P(\lambda)$. The *elementary divisors* of $P(\lambda)$ associated with the finite eigenvalue α_k are the collection of factors $(\lambda - \alpha_k)^{\delta_{jk}}$, including repetitions.

We say that $\lambda = \infty$ is an eigenvalue of a matrix polynomial $P(\lambda)$ if 0 is an eigenvalue of rev $P(\lambda) := \lambda^d P(1/\lambda)$. The elementary divisors λ^{γ_k} , where $\gamma_k > 0$, for the eigenvalue 0 of rev $P(\lambda)$ are the elementary divisors associated with ∞ of $P(\lambda)$.

For an $m \times n$ matrix polynomial $P(\lambda)$ define

$$\mathcal{N}_{\text{left}}(P(\lambda)) \coloneqq \{ y(\lambda)^T \in \mathbb{C}(\lambda)^{1 \times m} : y(\lambda)^T P(\lambda) = 0^T \} \text{ and } \\ \mathcal{N}_{\text{right}}(P(\lambda)) \coloneqq \{ x(\lambda) \in \mathbb{C}(\lambda)^{n \times 1} : P(\lambda) x(\lambda) = 0 \}$$

to be its left and right null-spaces, respectively, over the field $\mathbb{C}(\lambda)$. Every subspace \mathcal{W} of $\mathbb{C}(\lambda)^n$ has bases consisting entirely of vector polynomials. A basis of \mathcal{W} consisting of vector polynomials whose sum of degrees is minimal among all bases of \mathcal{W} consisting of vector polynomials is a minimal basis of \mathcal{W} . The minimal indices of \mathcal{W} are the degrees of the vector polynomials in a minimal basis of \mathcal{W} (they do not depend on the choice of a minimal basis). More formally, let the sets $\{y_1(\lambda)^T, ..., y_{m-r}(\lambda)^T\}$ and $\{x_1(\lambda), ..., x_{n-r}(\lambda)\}$ be minimal bases of $\mathcal{N}_{\text{left}}(P(\lambda))$ and $\mathcal{N}_{\text{right}}(P(\lambda))$, respectively, ordered so that $0 \leq \deg(y_1) \leq ... \leq \deg(y_{m-r})$ and $0 \leq \deg(x_1) \leq ... \leq \deg(x_{n-r})$. Let $\eta_k = \deg(y_k)$ for i = 1, ..., m - r and $\varepsilon_k = \deg(x_k)$ for i = 1, ..., n - r. Then the integers $0 \leq \eta_1 \leq \eta_2 \leq ... \leq \eta_{m-r}$ and $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq ... \leq \varepsilon_{n-r}$ are the left and right minimal indices of $P(\lambda)$, respectively. Note also that for a skewsymmetric matrix polynomial we have that $x_i(\lambda) = y_i(\lambda), i = 1, ..., n - r$ and thus $\eta_i = \varepsilon_i, i = 1, ..., n - r$.

We recall the following result from [10] which describes all possible combinations of the elementary divisors and minimal indices for a matrix polynomial of certain degree.

Theorem 3. [10] Let m, n, d, and r, such that $r \leq \min\{m, n\}$, be given positive integers. Let $g_1(\lambda), g_2(\lambda), \ldots, g_r(\lambda)$ be r arbitrarily monic polynomials with coefficients in \mathbb{C} and with respective degrees $\delta_1, \delta_2, \ldots, \delta_r$, and such that $g_j(\lambda)$ divides $g_{j+1}(\lambda)$ for $j = 1, \ldots, r-1$. Let $0 \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_r$, $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \ldots \leq \varepsilon_{n-r}$, and $0 \leq \eta_1 \leq \eta_2 \leq \ldots \leq \eta_{m-r}$ be given lists of integers. There exist an $m \times n$ matrix polynomial $P(\lambda)$ with rank r, degree d, invariant polynomials $g_1(\lambda), g_2(\lambda), \ldots, g_r(\lambda)$, with partial multiplicities at ∞ equal to $\gamma_1, \gamma_2, \ldots, \gamma_r$, and with right and left minimal indices equal to $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-r}$ and $\eta_1, \eta_2, \ldots, \eta_{m-r}$, respectively, if and only if

$$\sum_{j=1}^{r} \delta_j + \sum_{j=1}^{r} \gamma_j + \sum_{j=1}^{n-r} \varepsilon_j + \sum_{j=1}^{m-r} \eta_j = dr$$
(5)

holds and $\gamma_1 = 0$.

The condition $\gamma_1 = 0$ in Theorem 3 appears since we consider polynomials of exact degree d ($A_d \neq 0$). Using the definitions of elementary divisors and minimal indices we have the following lemma.

Lemma 4. Let $P(\lambda)$ be an $m \times n$ matrix polynomial with rank r, degree d, invariant polynomials $g_1(\lambda), g_2(\lambda), \ldots, g_r(\lambda)$, with partial multiplicities at ∞ equal to $\gamma_1, \gamma_2, \ldots, \gamma_r$, and with right and left minimal indices equal to $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-r}$ and $\eta_1, \eta_2, \ldots, \eta_{m-r}$, respectively, then the $n \times$ m matrix polynomial $P(\lambda)^T$ has rank r, degree d, invariant polynomials $g_1(\lambda), g_2(\lambda), \ldots, g_r(\lambda)$, partial multiplicities at ∞ equal to $\gamma_1, \gamma_2, \ldots, \gamma_r$, and right and left minimal indices equal to $\eta_1, \eta_2, \ldots, \eta_{m-r}$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-r}$, respectively.

Theorem 3 and Lemma 4 lead to the following theorem for skewsymmetric matrix polynomials.

Theorem 5. Let n, d, and r, such that $2r \leq n$, be given positive integers. Let $g_1(\lambda), g_2(\lambda), \ldots, g_r(\lambda)$ be r arbitrarily monic polynomials with coefficients in \mathbb{C} and with respective degrees $\delta_1, \delta_2, \ldots, \delta_r$, and such that $g_j(\lambda)$ divides $g_{j+1}(\lambda)$ for $j = 1, \ldots, r-1$. Let $0 \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_r$, and $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \ldots \leq \varepsilon_{n-2r}$ be given lists of integers. There exists a skew-symmetric $n \times n$ matrix polynomial $P(\lambda)$ with rank 2r, degree d, invariant polynomials $g_1(\lambda), g_1(\lambda), g_2(\lambda), g_2(\lambda), \ldots, g_r(\lambda), g_r(\lambda)$, with partial multiplicities at ∞ equal to $\gamma_1, \gamma_1, \gamma_2, \gamma_2, \ldots, \gamma_r, \gamma_r$, and with the right minimal indices equal to the left minimal indices, and equal to $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-2r}$ if and only if

$$\sum_{j=1}^{r} \delta_j + \sum_{j=1}^{r} \gamma_j + \sum_{j=1}^{n-2r} \varepsilon_j = dr$$
(6)

holds and $\gamma_1 = 0$.

Proof. For skew-symmetric polynomials the elementary divisors are coming in pairs, see Theorem 2, and the right minimal indices must be equal to the left minimal indices, see Lemma 4. Thus the equality (5) from Theorem 3 can be rewritten as

$$2\sum_{j=1}^r \delta_j + 2\sum_{j=1}^r \gamma_j + 2\sum_{j=1}^{n-2r} \varepsilon_j = 2dr.$$

Vice versa: Assume that (6) holds and $\gamma_1 = 0$ then by Theorem 3 (see also [31, Theorem 5.2]) there exist an $r \times (n-r)$ matrix polynomial $P(\lambda)$ with rank r, degree d ($A_d \neq 0$), invariant polynomials $g_1(\lambda), g_2(\lambda), \ldots, g_r(\lambda)$, with partial multiplicities at ∞ equal to $\gamma_1, \gamma_2, \ldots, \gamma_r$, and with right minimal indices equal to $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-2r}$. Therefore, by Lemma 4 the $(n-r) \times r$ matrix polynomial $P(\lambda)^T$ has rank r, degree d, invariant polynomials $g_1(\lambda), g_2(\lambda), \ldots, g_r(\lambda)$, partial multiplicities at ∞ equal to $\gamma_1, \gamma_2, \ldots, \gamma_r$, and left minimal indices equal to $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-2r}$. Since $\mathcal{N}_{\text{left}}(P(\lambda)) = \{0\}$, the vector $x(\lambda) \in \mathbb{C}(\lambda)^{(n-r)\times 1}$ is in the right null-space of $P(\lambda)$, i.e. $\mathcal{N}_{\text{right}}(P(\lambda))$, if and only if $\begin{bmatrix} 0\\ x(\lambda) \end{bmatrix} \in \mathbb{C}(\lambda)^{n \times 1}$ is in the right null-space of $W(\lambda)$, i.e. $\mathcal{N}_{\text{right}}(W(\lambda))$, where

$$W(\lambda) = \begin{bmatrix} 0 & P(\lambda) \\ -P(\lambda)^T & 0 \end{bmatrix}$$

Clearly, the analogous statement for the left null-spaces is also true. Therefore the $n \times n$ matrix polynomial $W(\lambda)$ is of degree d, has nonzero leading matrix-coefficient, and the required canonical structure information.

Now we know which combinations of the elementary divisors and minimal indices skew-symmetric matrix polynomials of certain degree may have.

4 Linearization of skew-symmetric matrix polynomials

A matrix pencil \mathcal{L} is called a *linearization* of a matrix polynomial $P(\lambda)$ if they have the same finite elementaty divisors. If in addition, rev \mathcal{L} is a linearization of rev $P(\lambda)$ then \mathcal{L} is called a *strong linearization* of $P(\lambda)$ [1, 36].

From this point we restrict to skew-symmetric matrix polynomials of odd degrees. The reason is that there is no linearization-template for skewsymmetric matrix polynomials of even degrees (actually, for the singular skew-symmetric matrix polynomials of even degrees linearizations do not exist at all) [37].

The following form is known to be a strong linearization of skewsymmetric $n \times n$ matrix polynomials $P(\lambda)$ of odd degrees [37], see also [1, 36]:

$$\mathcal{L}_{P(\lambda)}(i,i) = \begin{cases} \lambda A_{d-i+1} + A_{d-i} & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even,} \end{cases}$$
$$\mathcal{L}_{P(\lambda)}(i,i+1) = \begin{cases} -I_n & \text{if } i \text{ is odd,} \\ -\lambda I_n & \text{if } i \text{ is even,} \end{cases} \mathcal{L}_{P(\lambda)}(i+1,i) = \begin{cases} I_n & \text{if } i \text{ is odd,} \\ \lambda I_n & \text{if } i \text{ is even,} \end{cases}$$

where $\mathcal{L}_{P(\lambda)}(j,k)$ denotes an $n \times n$ matrix pencil which is at the position (j,k) of the block pencil $\mathcal{L}_{P(\lambda)}$ and $j,k = 1,\ldots,d$. We rewrite this strong

linearization template in a matrix form as follows

$$\mathcal{L}_{P(\lambda)} = \begin{bmatrix}
\lambda A_d + A_{d-1} & -I & & & \\
I & 0 & -\lambda I & & & \\
& \lambda I & \ddots & \ddots & & & \\
& & \ddots & 0 & -\lambda I & & \\
& & & \lambda I & \lambda A_3 + A_2 & -I & \\
& & & & I & 0 & -\lambda I \\
& & & & & \lambda I & \lambda A_1 + A_0
\end{bmatrix}$$
(7)

or

$$\mathcal{L}_{P(\lambda)} = \lambda \begin{bmatrix} A_d & & & \\ & \ddots & \ddots & & \\ & \ddots & 0 & -I & \\ & & I & A_3 & \\ & & & 0 & -I \\ & & & & I & A_1 \end{bmatrix} - \begin{bmatrix} -A_{d-1} & I & & & \\ & -I & 0 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & -A_2 & I & \\ & & & & -I & 0 & \\ & & & & & -A_0 \end{bmatrix}.$$
(8)

Strong linearizations preserve finite and infinite elementary divisors but do not usually preserve the left and right minimal indices. Nevertheless, the relations between the minimal indices of matrix polynomials and their Fiedler linearizations are derived in [8, 9]. We apply these results to describe the changes of the minimal indices in our case.

Theorem 6. Let $P(\lambda)$ be a skew-symmetric $n \times n$ matrix polynomial of odd degree $d \ge 3$, and let $\mathcal{L}_{P(\lambda)}$ be its linerization (7) given above. If $0 \le \varepsilon_1 \le \varepsilon_2 \le \ldots \le \varepsilon_t$ are the right (=left) minimal indices of $P(\lambda)$ then

$$0 \leq \varepsilon_1 + \frac{1}{2}(d-1) \leq \varepsilon_2 + \frac{1}{2}(d-1) \leq \ldots \leq \varepsilon_t + \frac{1}{2}(d-1)$$

are the right (=left) minimal indices of $\mathcal{L}_{P(\lambda)}$.

Proof. First, note that

$$\mathcal{L}_{P(\lambda)} \coloneqq \operatorname{diag}(I_n, -I_n, I_n, -I_n, \dots, (-1)^{d-1}I_n)(\lambda \mathcal{L}_{odd}^{-1} - \mathcal{L}_{even}),$$

where $\mathcal{L}_{even} = L_0 L_2 \dots L_{d-1}$ and $\mathcal{L}_{odd} = L_1 L_3 \dots L_{d-2} L_d^{-1}$ with

$$L_{d} := \begin{bmatrix} A_{d} & & \\ & I_{(d-1)n} \end{bmatrix}, \qquad L_{0} := \begin{bmatrix} I_{(d-1)n} & & \\ & -A_{0} \end{bmatrix}, \quad \text{and}$$
$$L_{i} := \begin{bmatrix} I_{(d-i-1)n} & & & \\ & -A_{i} & I_{n} & \\ & & I_{n} & 0 & \\ & & & & I_{(i-1)n} \end{bmatrix}, \qquad i = 1, \dots, d-1.$$

Therefore $\mathcal{L}_{P(\lambda)}$ is strictly equivalent to

$$\lambda L_d L_{d-2}^{-1} \dots L_3^{-1} L_1^{-1} - L_0 L_2 \dots L_{d-1}$$

which is, in turn, strictly equivalent to the following Fiedler linerization

$$\mathcal{F}_{P(\lambda)} = \lambda L_d - L_0 L_2 \dots L_{d-1} L_1 L_3 \dots L_{d-2}.$$

The order of multipliers in the second (constant) matrix of the pencil $\mathcal{F}_{P(\lambda)}$ can be determined by the following bijection associated with $\mathcal{F}_{P(\lambda)}$

$$\sigma: \{0, 1, \dots, d-1\} \to \{1, \dots, d\}; \quad \sigma(i) = \frac{i}{2} + 1 + \frac{d}{4}(1 - (-1)^i),$$

with inverse σ^{-1} , i.e., we have

$$L_{\sigma^{-1}(1)}L_{\sigma^{-1}(2)}\ldots L_{\sigma^{-1}(d)} = L_0L_2\ldots L_{d-1}L_1L_3\ldots L_{d-2}$$

Recall that due to the skew symmetry, the right minimal indices are equal to the left minimal indices but here we prefer to use ε_i for the right and η_i for the left minimal indices ($\varepsilon_i = \eta_i$), since they will be changed differently (resulting, nevertheless, in the same value). By [8] we have that the right and left minimal indices of $\mathcal{F}_{P(\lambda)}$, and thus of $\mathcal{L}_{P(\lambda)}$, will be

$$0 \leq \varepsilon_1 + i(\sigma) \leq \varepsilon_2 + i(\sigma) \leq \ldots \leq \varepsilon_t + i(\sigma)$$

and

$$0 \leq \eta_1 + c(\sigma) \leq \eta_2 + c(\sigma) \leq \ldots \leq \eta_t + c(\sigma),$$

where $i(\sigma)$ and $c(\sigma)$ are the total numbers of inversions and consecutions in σ , see [8] for more details. Now the remaining part is to note that for $i = 0, \ldots, d-2$ we have $\sigma(i) < \sigma(i+1)$, i.e., consecution, if *i* is even, and $\sigma(i) > \sigma(i+1)$, i.e., inversion, if *i* is odd. Therefore $i(\sigma) = c(\sigma) = \frac{1}{2}(d-1)$. \Box The "shifts" of the minimal indices described in Theorem 6 show that the linearization of a skew-symmetric matrix polynomial (which is a skewsymmetric matrix pencil with a special block-structure) may have singular M_m blocks (see Theorem 1) only of the sizes greater than or equal to d. More formally, each pair of minimal indices ε_j and η_j ($\varepsilon_j = \eta_j$) of $P(\lambda)$ is "shifted" and results in a singular M_m block of the linearization $\mathcal{L}_{P(\lambda)}$ with $\varepsilon_j + \frac{1}{2}(d-1) + \eta_j + \frac{1}{2}(d-1) + 1 = \varepsilon_j + \eta_j + d = 2\varepsilon_j + d$ rows and the same number of columns.

Let us describe which congruence orbits of skew-symmetric $dn \times dn$ matrix pencils contain pencils that are the linearizations of some skew-symmetric $n \times n$ matrix polynomials of odd degree d. By Theorem 5 for a skew-symmetric matrix polynomial $P(\lambda)$ with the finite elementary divisors $\delta_1, \delta_2, \ldots, \delta_r$, the infinite elementary divisors $\gamma_1, \gamma_2, \ldots, \gamma_r$, and the left minimal indices, equal to the right minimal indices, and equal to $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-2r}$ we have

$$2\sum_{j=1}^{r} \delta_j + 2\sum_{j=1}^{r} \gamma_j + 2\sum_{j=1}^{n-2r} \varepsilon_j = 2dr.$$

Adding (n-2r)d to both sides we obtain

$$2\sum_{j=1}^{r} \delta_j + 2\sum_{j=1}^{r} \gamma_j + 2\sum_{j=1}^{n-2r} \varepsilon_j + (n-2r)d = 2dr + (n-2r)d,$$

or equivalently

$$2\sum_{j=1}^{r} \delta_j + 2\sum_{j=1}^{r} \gamma_j + \sum_{j=1}^{n-2r} \left(2\varepsilon_j + d \right) = dn, \tag{9}$$

where the left hand side is exactly the sum of the sizes of the Jordan blocks H and K, and singular blocks M of the canonical form of $\mathcal{L}_{P(\lambda)}$. Summing up, a skew-symmetric $dn \times dn$ matrix pencil is congruent to a pencil that is the linearization of a skew-symmetric $n \times n$ matrix polynomial of degree d if and only if the canonical structure information of this pencil satisfies (9).

5 Versal deformations of matrix polynomial linearizations

Recall that our goal is to investigate changes under small perturbations of the canonical structure information of skew-symmetric $n \times n$ matrix polynomials

of degree d, by studying perturbations of the $dn \times dn$ matrix pencils that are the linearizations of these polynomials. In this section, using so called versal deformations, we prove that it is enough to perturb only those blocks of the linearizations that are the coefficient matrices of matrix polynomial, see Theorem 8. Exploring essentially the same ideas, the analogous result for the first and the second companion forms is proven in [31].

The notion of a *(mini)versal deformation* of a matrix with respect to similarity was introduced by V.I. Arnold [2] (see also [3, Ch. 30B]). Later this notion has been extended to general matrix pencils [23, 27], as well as to matrix pencils with symmetries [12, 13, 14, 15] and, in particular, skew-symmetric matrix pencils [12]. Recall that: a *deformation* of a skewsymmetric $n \times n$ matrix pencil R is a holomorphic mapping $R(\vec{\sigma})$, where $\vec{\sigma} := (\sigma_1, \ldots, \sigma_k)$, from a neighbourhood $\Omega \subset \mathbb{C}^k$ of $\vec{0} = (0, \ldots, 0)$ to the space of $n \times n$ matrix pencils such that $R(\vec{0}) = R$. A deformation $R(\sigma_1, \ldots, \sigma_k)$ of a matrix pencil R is called *versal* if for every deformation $Q(\delta_1, \ldots, \delta_l)$ of Rwe have

$$Q(\delta_1,\ldots,\delta_l) = \mathcal{I}(\delta_1,\ldots,\delta_l)^T R(\varphi_1(\vec{\delta}),\ldots,\varphi_k(\vec{\delta})) \mathcal{I}(\delta_1,\ldots,\delta_l),$$

where $\mathcal{I}(\delta_1, \ldots, \delta_l)$ is a deformation of the identity matrix, and all $\varphi_i(\vec{\delta})$ are convergent in a neighborhood of $\vec{0}$ power series such that $\varphi_i(\vec{0}) = 0$. A versal deformation $R(\sigma_1, \ldots, \sigma_k)$ of R is called *miniversal* if there is no versal deformation having less than k parameters. Informally speaking, a versal deformation is a normal form to which all matrices close to a given matrix can be smoothly reduced.

We investigate all matrix pencils in a neighbourhood of $\mathcal{L}_{P(\lambda)}$, i.e.,

$$\mathcal{L}_{P(\lambda)} + E = \lambda \begin{bmatrix}
A_{d} & & & \\
& \ddots & \ddots & \\
& I & A_{3} & \\
& & I & A_{3} & \\
& & & I & A_{1}
\end{bmatrix} - \begin{bmatrix}
-A_{d-1} & I & & \\
-I & 0 & \ddots & & \\
& & \ddots & \ddots & \\
& & & -A_{2} & I & \\
& & & -I & 0 & \\
& & & & -A_{0}
\end{bmatrix}$$

$$+ \lambda \begin{bmatrix}
\widetilde{E}_{11} & \widetilde{E}_{12} & \widetilde{E}_{13} & \dots & \widetilde{E}_{1d} \\
\widetilde{E}_{21} & \widetilde{E}_{22} & \widetilde{E}_{23} & \dots & \widetilde{E}_{2d} \\
\widetilde{E}_{31} & \widetilde{E}_{32} & \widetilde{E}_{33} & \dots & \widetilde{E}_{3d} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\widetilde{E}_{d1} & \widetilde{E}_{d2} & \widetilde{E}_{d3} & \dots & \widetilde{E}_{dd}
\end{bmatrix} - \begin{bmatrix}
\widetilde{E}_{11} & \widehat{E}_{12} & \widehat{E}_{13} & \dots & \widehat{E}_{1d} \\
\widetilde{E}_{21} & \widehat{E}_{22} & \widehat{E}_{23} & \dots & \widehat{E}_{2d} \\
\widetilde{E}_{31} & \widehat{E}_{32} & \widehat{E}_{33} & \dots & \widehat{E}_{3d} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\widetilde{E}_{d1} & \widehat{E}_{d2} & \widehat{E}_{d3} & \dots & \widehat{E}_{dd}
\end{bmatrix},$$
(10)

where $E = \lambda[\tilde{E}_{ij}] - [\hat{E}_{ij}]$ is skew symmetric and has arbitrarily small entries. In particular, we allow perturbations of the zero and identity blocks in $\mathcal{L}_{P(\lambda)}$ and thus the form of $\mathcal{L}_{P(\lambda)}$ is not required to be preserved. Our goal is to find a matrix pencil $\mathcal{L}_{P(\lambda)}(E)$ to which all $dn \times dn$ matrix pencils $\mathcal{L}_{P(\lambda)} + E$ that are close to a given $\mathcal{L}_{P(\lambda)}$, can be reduced by

$$\mathcal{L}_{P(\lambda)} + E \mapsto W(E)^T (\mathcal{L}_{P(\lambda)} + E) W(E) =: \mathcal{L}_{P(\lambda)}(E),$$
(11)

where W(E) is holomorphic at 0 (i.e., its entries are power series in the entries of E that are convergent in a neighborhood of 0), W(0) is a nonsingular matrix. By choosing W(0) to be identity and (11), we have $\mathcal{L}_{P(\lambda)}(0)$ equal to $\mathcal{L}_{P(\lambda)}$. Define a matrix pencil $\mathcal{D}(E)$ by

$$\mathcal{L}_{P(\lambda)} + \mathcal{D}(E) = W(E)^T (\mathcal{L}_{P(\lambda)} + E) W(E).$$
(12)

Therefore $\mathcal{D}(E)$ is holomorphic at 0 and $\mathcal{D}(0) = 0$. Similarly to [12, 13, 14, 15, 23], we have that $\mathcal{L}_{P(\lambda)} + \mathcal{D}(E)$ is a versal deformation of $\mathcal{L}_{P(\lambda)}$.

Following the notation of [14], denote by $\mathcal{D}(\mathbb{C})$ the space of all skewsymmetric matrix pencils obtained from $\mathcal{D}(E)$ by replacing its nonzero entries by complex numbers:

$$\mathcal{D}(\mathbb{C}) \coloneqq \lambda \left(+ \bigcup_{(i,j) \in \mathcal{I} nd_1(\mathcal{D})} \mathbb{C} V_{ij} \right) - \left(+ \bigcup_{(i,j) \in \mathcal{I} nd_2(\mathcal{D})} \mathbb{C} V_{ij} \right),$$
(13)

where

$$\mathcal{I}nd_1(\mathcal{D}), \mathcal{I}nd_2(\mathcal{D}) \subseteq \{1, \dots, dn\} \times \{1, \dots, dn\}$$
 (14)

are the sets of indices of the nonzero entries in the upper-triangular parts of the first and the second matrices, respectively, of the pencil $\mathcal{D}(E)$, and V_{ij} is the matrix whose (i, j)-th entry is 1, (j, i)-th entry is -1, and the other entries are 0s. Note that "+" denotes the entrywise sum of matrices.

Define $\mathbb{C}_{\text{skew}}^{n \times n}$ to be a space of complex skew-symmetric $n \times n$ matrices. We recall the following lemma which is also presented in [3, 4, 12, 14, 23].

Lemma 7. Let $\mathcal{L}_{P(\lambda)} \in \mathbb{C}^{dn \times dn}_{skew} \times \mathbb{C}^{dn \times dn}_{skew}$ be of the form (7) and $\mathcal{D}(E) \in \mathbb{C}^{dn \times dn}_{skew} \times \mathbb{C}^{dn \times dn}_{skew}$. The deformation $\mathcal{L}_{P(\lambda)} + \mathcal{D}(E)$ is versal if and only if the vector space $\mathbb{C}^{dn \times dn}_{skew} \times \mathbb{C}^{dn \times dn}_{skew}$ decomposes into the sum $T^{c}_{\mathcal{L}_{P(\lambda)}} + \mathcal{D}(\mathbb{C})$, where $T^{c}_{\mathcal{L}_{P(\lambda)}}$ is the tangent space to the congruence orbit of $\mathcal{L}_{P(\lambda)}$, see (2), at the point $\mathcal{L}_{P(\lambda)}$ and $\mathcal{D}(\mathbb{C})$ is defined in (13).

Proof. In a small neighbourhood of the point $\mathcal{L}_{P(\lambda)}$ only linear deformations matter and the curvature of the orbit becomes unimportant (see [3, Sec. 1.6] or [2, 23]). This allows us to "associate" the orbit of $\mathcal{L}_{P(\lambda)}$ with its tangent space at the point $\mathcal{L}_{P(\lambda)}$. Therefore a versal deformation of $\mathcal{L}_{P(\lambda)}$ is transversal to $T_{\mathcal{L}_{P(\lambda)}}$ (two subspaces of a vector space are called *transversal* if their sum is equal to the whole space [4, Ch. 29]).

Theorem 8 presents a versal deformation of $\mathcal{L}_{P(\lambda)}$ where only the blocks that are the coefficient matrices of $P(\lambda)$ are perturbed. This form of the versal deformations ensures Theorem 10 that will be crucial in Section 7.

Theorem 8. Let $\mathcal{L}_{P(\lambda)}$ be a skew-symmetric matrix pencil of the form (7). Its versal deformation can be taken in the form

$$\mathcal{L}_{P(\lambda)} + F$$

$$= \lambda \begin{bmatrix} A_d & & & & \\ & \ddots & \ddots & & \\ & & I & A_3 & & \\ & & & I & A_3 & \\ & & & & I & A_1 \end{bmatrix} - \begin{bmatrix} -A_{d-1} & I & & & & \\ -I & 0 & \ddots & & & \\ & \ddots & \ddots & & & \\ & & & -A_2 & I & \\ & & & & -I & 0 & \\ & & & & -I & 0 & \\ & & & & -I & 0 & \\ & & & & & -A_0 \end{bmatrix}$$
(15)
$$+ \lambda \begin{bmatrix} F_d & 0 & \dots & 0 \\ 0 & \ddots & \ddots & & \\ & \ddots & 0 & 0 & \vdots \\ \vdots & 0 & F_3 & 0 & \\ & & & 0 & 0 & 0 \\ 0 & \dots & 0 & F_1 \end{bmatrix} - \begin{bmatrix} F_{d-1} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & & \\ & \ddots & 0 & 0 & \vdots \\ \vdots & 0 & F_2 & 0 & \\ & & & 0 & 0 & 0 \\ 0 & \dots & 0 & F_0 \end{bmatrix}$$

in which $F_i, i \in \{0, 1, ..., d\}$ are matrices with arbitrarily small entries (all the entries are independent from each other).

Proof. We will show that the statement holds for d = 3. The general proof is

analogous. The tangent space to the congruence orbit of $\mathcal{L}_{P(\lambda)}$ (see (2)) is

$$\mathbf{T}_{\mathcal{L}_{P(\lambda)}}^{c} = \left\{ \lambda \begin{pmatrix} \begin{bmatrix} C_{11}^{T} & C_{21}^{T} & C_{31}^{T} \\ C_{12}^{T} & C_{22}^{T} & C_{32}^{T} \\ C_{13}^{T} & C_{23}^{T} & C_{33}^{T} \end{bmatrix} \begin{bmatrix} A_{3} & 0 & 0 \\ 0 & 0 & -I \\ 0 & I & A_{1} \end{bmatrix} + \begin{bmatrix} A_{3} & 0 & 0 \\ 0 & 0 & -I \\ 0 & I & A_{1} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \right)$$
(16)

$$-\left(\begin{bmatrix}C_{11}^{T} & C_{21}^{T} & C_{31}^{T}\\C_{12}^{T} & C_{22}^{T} & C_{32}^{T}\\C_{13}^{T} & C_{23}^{T} & C_{33}^{T}\end{bmatrix}\begin{bmatrix}-A_{2} & I & 0\\-I & 0 & 0\\0 & 0 & -A_{0}\end{bmatrix}+\begin{bmatrix}-A_{2} & I & 0\\-I & 0 & 0\\0 & 0 & -A_{0}\end{bmatrix}\begin{bmatrix}C_{11} & C_{12} & C_{13}\\C_{21} & C_{22} & C_{23}\\C_{31} & C_{32} & C_{33}\end{bmatrix}\right)\right\}$$
(17)

$$= \left\{ \lambda \begin{bmatrix} C_{11}^{T}A_{3} + A_{3}C_{11} & A_{3}C_{12} + C_{31}^{T} & C_{31}^{T}A_{1} + A_{3}C_{13} - C_{21}^{T} \\ C_{12}^{T}A_{1} - C_{31} & C_{32}^{T} - C_{32} & C_{32}^{T}A_{1} - C_{22}^{T} - C_{33} \\ C_{13}^{T}A_{3} + A_{1}C_{31} + C_{21} & A_{1}C_{32} + C_{33}^{T} + C_{22} & C_{33}^{T}A_{3} + A_{3}C_{33} + C_{23} - C_{23}^{T} \end{bmatrix}$$
(18)

$$-\begin{bmatrix} -C_{11}^{T}A_{2} - A_{2}C_{11} + C_{21} - C_{21}^{T} & -A_{2}C_{12} + C_{22} + C_{11}^{T} & -C_{31}^{T}A_{0} - A_{2}C_{13} + C_{23} \\ -C_{12}^{T}A_{2} - C_{22}^{T} - C_{11} & C_{12}^{T} - C_{12} & -C_{32}^{T}A_{0} - C_{13} \\ -C_{13}^{T}A_{2} - A_{0}C_{31} - C_{23}^{T} & -A_{0}C_{32} + C_{13}^{T} & -C_{33}^{T}A_{0} - A_{0}C_{33} \end{bmatrix} \right\}.$$
(19)

Since C_{ij} , i, j = 1, 2, 3 are arbitrarily, the subblocks (1, 2), (1, 3), (2, 2), and (2, 3) in the pencil (18)–(19) can get any values, regardless to the values of the matrices A_i , i = 0, 1, 2, 3. Note that the blocks at the positions (2, 1), (3, 1), and (3, 2) of (18)–(19) are the negated and transposed blocks at the positions (1, 2), (1, 3), and (2, 3) of (18)–(19), respectively. Therefore the subspace

$$\mathcal{D}(\mathbb{C}) = \left\{ F = \lambda \begin{bmatrix} F_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F_1 \end{bmatrix} - \begin{bmatrix} F_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F_0 \end{bmatrix} \right\}$$
(20)

where $F_i, i = 0, ..., 3$ are any skew-symmetric matrices of conforming sizes, is transversal to $T^c_{\mathcal{L}_{P(\lambda)}}$. By Lemma 7, $\mathcal{L}_{P(\lambda)} + F$ (15) $(F_0, ..., F_d$ in F have arbitrarily small entries) is a versal deformation of $\mathcal{L}_{P(\lambda)}$. Note that $\mathcal{D}(\mathbb{C})$ is transversal to $T^c_{\mathcal{L}_{P(\lambda)}}$ but not of minimal dimension, thus the deformation $\mathcal{L}_{P(\lambda)} + F$ is versal but not miniversal.

6 Linearization orbits and bundles of skewsymmetric matrix polynomials and their codimensions

In Section 5, we considered the linearization $\mathcal{L}_{P(\lambda)}$ (7) under congruence which preserves skew symmetry but does not preserve the block structure of $\mathcal{L}_{P(\lambda)}$. Therefore many elements of $O^{c}_{\mathcal{L}_{P(\lambda)}}$ (1) are not the linearizations of any skew-symmetric matrix polynomial. This motivates us to define $O_{\mathcal{L}_{P(\lambda)}}$ that consists only of skew-symmetric matrix pencils that are the linearizations of skew-symmetric matrix polynomials.

Define the generalized sylvester space for $P(\lambda)$ as follows

$$GSYL_{skew}(\mathcal{L}_{P(\lambda)}) = \{\mathcal{L}_{P(\lambda)} : P(\lambda) \text{ are skew-symmetric} \\ n \times n \text{ matrix polynomials}\}.$$
(21)

If there is no risk of confusion we will write $\text{GSYL}_{\text{skew}}$ instead of $\text{GSYL}_{\text{skew}}(\mathcal{L}_{P(\lambda)})$. Now we define orbits of the linearizations of matrix polynomials

$$O_{\mathcal{L}_{P(\lambda)}} = \{ (R^T \mathcal{L}_{P(\lambda)} R) \in GSYL_{skew}(\mathcal{L}_{P(\lambda)}) : R \in GL_n(\mathbb{C}) \}.$$
(22)

By [31, Lemma 9.2] we have that $O_{\mathcal{L}_{P(\lambda)}}$ is a manifold in the matrix pencil space. Recall that $\dim O_{\mathcal{L}_{P(\lambda)}}^c := \dim T_{\mathcal{L}_{P(\lambda)}}^c$ and $\dim O_{\mathcal{L}_{P(\lambda)}} := \dim (GSYL_{skew} \cap T_{\mathcal{L}_{P(\lambda)}}^c)$, respectively, and the dimensions of the corresponding normal spaces are equal to the codimensions of the orbits. Notably, codimensions of the orbits give a coarse stratification: only orbits with higher codimensions may be in the closure of a given orbit. The codimensions for skew-symmetric matrix pencils were computed in [21] and implemented in the MCS Toolbox [17]. These codimensions are also equal to the number of independent parameters in the miniversal deformations from [12]. The following theorem shows that the codimensions of the congruence orbits of skew-symmetric matrix pencils, i.e. $\operatorname{cod} O_{\mathcal{L}_{P(\lambda)}}^c$, are equal to the codimensions of the orbits of the linearization of skew-symmetric matrix polynomials, i.e. $\operatorname{cod} O_{\mathcal{L}_{P(\lambda)}}$.

Theorem 9. Let $\mathcal{L}_{P(\lambda)}$ be a matrix pencil of the form (7). Then $\operatorname{cod} O_{\mathcal{L}_{P(\lambda)}}^{c} = \operatorname{cod} O_{\mathcal{L}_{P(\lambda)}}^{c}$.

Proof. Note that $\mathbb{C}^{dn \times dn}_{skew} \times \mathbb{C}^{dn \times dn}_{skew}$ is the least affine space containing $T^{c}_{\mathcal{L}_{P(\lambda)}}$ and $GSYL_{skew}$ (see Theorem 8), and since $T^{c}_{\mathcal{L}_{P(\lambda)}} \cap GSYL_{skew} \neq \emptyset$ we have

$$\dim(\mathbb{C}^{dn\times dn}_{\text{skew}} \times \mathbb{C}^{dn\times dn}_{\text{skew}}) = \dim T^{c}_{\mathcal{L}_{P(\lambda)}} + \dim \text{GSYL}_{\text{skew}} - \dim(\text{GSYL}_{\text{skew}} \cap T^{c}_{\mathcal{L}_{P(\lambda)}}).$$

Therefore

$$\begin{array}{l} \operatorname{cod} \operatorname{O}_{\mathcal{L}_{P(\lambda)}}^{c} = \dim(\mathbb{C}_{\operatorname{skew}}^{dn \times dn} \times \mathbb{C}_{\operatorname{skew}}^{dn \times dn}) - \dim \operatorname{O}_{\mathcal{L}_{P(\lambda)}}^{c} \\ = \dim \operatorname{T}_{\mathcal{L}_{P(\lambda)}}^{c} + \dim \operatorname{GSYL}_{\operatorname{skew}} - \dim(\operatorname{GSYL}_{\operatorname{skew}} \cap \operatorname{T}_{\mathcal{L}_{P(\lambda)}}^{c})) - \dim \operatorname{T}_{\mathcal{L}_{P(\lambda)}}^{c} \\ = \dim \operatorname{GSYL}_{\operatorname{skew}} - \dim \operatorname{O}_{\mathcal{L}_{P(\lambda)}} = \operatorname{cod} \operatorname{O}_{\mathcal{L}_{P(\lambda)}}. \end{array}$$

Define a bundle $B_{\mathcal{L}_{P(\lambda)}}$ of the matrix polynomial linearization $\mathcal{L}_{P(\lambda)}$ to be a union of the orbits $O_{\mathcal{L}_{P(\lambda)}}$ with the same singular structures and the same Jordan structures except that the distinct eigenvalues may be different. This definition was given in [31] and is the same as the one for (skew-symmetric) matrix pencils [20, 24]. Therefore, two linearizations $\mathcal{L}_{P(\lambda)}$ and $\mathcal{L}_{Q(\lambda)}$ are in the same bundle if and only if they are in the same bundle as skew-symmetric matrix pencils. The codimensions of the bundles of $\mathcal{L}_{P(\lambda)}$ are defined as

$$\operatorname{cod} B_{\mathcal{L}_{P(\lambda)}} = \operatorname{cod} O_{\mathcal{L}_{P(\lambda)}} - \# \left\{ \operatorname{distinct \ eigenvalues \ of \ } \mathcal{L}_{P(\lambda)} \right\}.$$

Bundles are useful in many applications, see for example [24, 25, 31, 33], where the eigenvalues of the underlying matrices may coalesce or split apart with the change of their values. More about bundles and their stratifications can be found in [16, 20, 23, 25, 31], see also Example 13.

7 Stratification of linearizations of skewsymmetric matrix polynomials

In this section, we present an algorithm for constructing the orbit and bundle stratifications of the linearizations for skew-symmetric matrix polynomials of odd degrees. This algorithm is similar to [20, Algorithm 4.1] for skew-symmetric matrix pencils.

First we show that all linearizations that are attainable by perturbations of the form (10) are also attainable by perturbations of the form (15).

Theorem 10. Let $P(\lambda)$ and $Q(\lambda)$ be two skew-symmetric $n \times n$ matrix polynomials of the same odd degree, and $\mathcal{L}_{P(\lambda)}$ and $\mathcal{L}_{Q(\lambda)}$ be their linearizations (7). There exists an arbitrarily small (entrywise) skew-symmetric perturbation E of the linearization $\mathcal{L}_{P(\lambda)}$, i.e. $\mathcal{L}_{P(\lambda)} + E$, and a nonsingular matrix C, such that

$$C^{T}(\mathcal{L}_{P(\lambda)} + E)C = \mathcal{L}_{Q(\lambda)}$$
(23)

if and only if there exists an arbitrarily small (entrywise) skew-symmetric perturbation $F(\lambda)$ of $P(\lambda)$, i.e. $P(\lambda) + F(\lambda)$, and a nonsingular matrix S, such that

$$S^{T}\mathcal{L}_{P(\lambda)+F(\lambda)}S = \mathcal{L}_{Q(\lambda)}.$$
(24)

Proof. The proof follows directly from Theorem 8 which states that each perturbation of the linearization of a skew-symmetric $n \times n$ matrix polynomial $\mathcal{L}_{P(\lambda)} + E$ can be smoothly reduced by congruence to the one in which only the blocks $A_i, i = 0, 1, \ldots$ are perturbed, i.e. $\mathcal{L}_{P(\lambda)} + \mathcal{D}(E)$ that is equal to $\mathcal{L}_{P(\lambda)+F(\lambda)}$ for some $F(\lambda)$.

Below we outline an algorithm for the orbit and bundle stratifications of the linearizations for skew-symmetric $n \times n$ matrix polynomials of odd degrees. The algorithm relies on the orbit and bundle stratifications of skewsymmetric matrix pencils [20] and Theorems 5, 6, and 10, that in turn use many other results.

Algorithm 11. Steps 1–3 produce the orbit (bundle) stratification of the linearization for skew-symmetric $n \times n$ matrix polynomials of odd degree d.

- **Step 1.** Construct the orbit (bundle) stratification of skew-symmetric dn×dn matrix pencils under congruence [20].
- **Step 2.** Extract from the stratification obtained at Step 1 the nodes that correspond to the linearizations of skew-symmetric $n \times n$ matrix polynomials of degree d (see Theorems 5 and 6).
- Step 3. Put an edge (arrow) between two nodes obtained at Step 2 if there is a path between the corresponding nodes obtained at Step 1; otherwise no edge is inserted (see Theorem 10).

By the following two examples, we illustrate Algorithm 11 as well as the difference in the orbit and bundle stratification graphs, e.g., see the numbers of the most generic nodes.

Example 12. In this example we stratify the orbit linearizations of skewsymmetric 2×2 matrix polynomials of degree 3. Using Theorem 5, we need to determine which combinations of the elementary divisors and minimal indices such polynomials may have: the condition (6) looks like $\delta_1 = 3$ (recall that the leading coefficient is nonzero). Therefore all such polynomials are regular. Note that $\delta_1 = 3$ is just the degree of the invariant polynomial which



Figure 1: Orbit stratification graph for skew-symmetric 6×6 matrix pencils. The three top-most nodes (in bold) form the orbit stratification graph (with no arrows) of the linearizations for skew-symmetric 2×2 matrix polynomials of degree 3.

gives three possibilities for the powers of elementary divisors, resulting in the following canonical forms for the considered linearizations: $H_3(\mu_1), H_2(\mu_1) \oplus H_1(\mu_2)$, and $H_1(\mu_1) \oplus H_1(\mu_2) \oplus H_1(\mu_3)$.

In Figure 1, we present the orbit stratification of skew-symmetric 6×6 matrix pencils constructed using the algorithm from [20]. The numbers to the right of the graph are the computed orbit codimensions [17, 21]. The subgraph that includes just the three most generic nodes (in bold) and no arrows is the orbit stratification of the linearizations of skew-symmetric 2×2



Figure 2: Bundle stratification graph for skew-symmetric 6×6 matrix pencils. The top-most three nodes and two arrows (in bold) form the bundle stratification of the linearizations for skew-symmetric 2×2 matrix polynomials of degree 3.

matrix polynomials of degree 3. The fact that we have no arrows means that if the values of the eigenvalues are fixed then small perturbations cannot change the canonical structure information of skew-symmetric 2×2 matrix polynomials of degree 3, i.e., the stratification consists of three unconnected graphs.

Example 13. Following Example 12, in Figure 2 we derive the bundle stratification of the linearizations for skew-symmetric 2×2 matrix polynomials of degree 3 by extracting it from the bundle stratification for skew-symmetric 6×6 matrix pencils (see [20]). The obtained graph consists of the three topmost nodes and two arrows (in bold). As in Example 12, the numbers to the right of the graph are the computed bundle codimensions [17, 21].

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