Canonical structure transitions of system pencils

by

Andrii Dmytryshyn, Stefan Johansson, and Bo Kågström

UMINF 15.15

UMEÅ UNIVERSITY
DEPARTMENT OF COMPUTING SCIENCE
SE-901 87 UMEÅ
SWEDEN

Canonical structure transitions of system pencils [⋆,⋆⋆]

Andrii Dmytryshyn [a], Stefan Johansson [a], and Bo Kågström [a]

[a] *Department of Computing Science, Umeå University, SE-901 87 Umeå, Sweden*

Abstract

We investigate the changes under small perturbations of the canonical structure information for a system pencil

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} - s \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad \det(E) \neq 0,$$

associated with a (generalized) linear time-invariant state-space system. The equivalence class of the pencil is taken with respect to feedback-injection equivalence transformation. The results allow to track possible changes under small perturbations of important linear system characteristics.

Key words: linear system; descriptor system; state-space system; system pencil; matrix pencil; orbit; bundle; perturbation; versal deformation; stratification.

1 Introduction

A common approach to determine the finite and infinite zeros as well as the singular structure of a linear system, is to compute the generalized eigenvalues and minimal indices (canonical structure information) of an associated system pencil. However, in general this is an ill-posed problem in the sense that small perturbations in the matrices may drastically change the computed system characteristics. To analyze how arbitrary small perturbations change the canonical structure Arnold introduced (mini)versal deformations for matrices [2]. In Section 3, we continue this work by considering versal deformations

⋆ Preprint Report UMINF 15.15, Dept. Computing Science, Umeå University
⋆⋆ Corresponding author A. Dmytryshyn.
Email addresses: andrii@cs.umu.se (Andrii Dmytryshyn), stefanj@cs.umu.se (Stefan Johansson), bokg@cs.umu.se (and Bo Kågström).

of system pencils associated with the *non-singular continuous-time generalized state-space system* (or *descriptor* system) of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \qquad \det(E) \neq 0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

where $A, E \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, $D \in \mathbb{C}^{p\times m}$, and $x(t), y(t), u(t)$ are the *state, output,* and *input (control) vectors* of conforming sizes. Obviously, this kind of system can be transformed into a standard state-space system by multiplying with E^{-1} from the left. However, since the results in this paper are not limited to the case $E = I$ and E may be close to singular, we preserve (1) in the generalized form. Sometimes we also refer to the associated *transfer function* $H(s)$, where $H(s)$ is a $p\times m$ rational matrix representing the system in frequency domain. The system (1) is called a *realization* of $H(s)$ if $H(s) = C(sE - A)^{-1}B + D, \det(E) \neq 0$, is satisfied. A realization is *minimal* if and only if it is both controllable and observable.

With system (1) we associate the following *system pencil* (blocked matrix pencil) of size $(n+p)\times(n+m)$

$$\mathcal{S} := \begin{bmatrix} A & B \\ C & D \end{bmatrix} - s \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad \det(E) \neq 0. \tag{2}$$

In addition, we investigate the special case $D \equiv 0$, i.e., the system (1) with no direct feedforward, using the system pencil

$$\mathcal{S}_0 := \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - s \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad \det(E) \neq 0, \tag{3}$$

which is a realization of a *strictly proper* transfer function, i.e., a transfer function with the degree of the nominator smaller than the degree of the denominator.

In line of previous work on matrix pencils [16], and especially controllability pencils $\begin{bmatrix} A - sI & B \end{bmatrix}$, observability pencils $\left[\begin{smallmatrix} A-sI \\ C \end{smallmatrix}\right]$ [17] and related works on system pencils [5,7,19], we present in Sections 4 and 5 the *stratification theory* of $\mathcal{S}$ and $\mathcal{S}_0$, which reveals how the canonical forms and associated system characteristics can change under arbitrarily small perturbations of the original system pencil. The stratification shows the closure hierarchy of orbits (and bundles) of canonical structures, where an orbit is a manifold of system pencils with the same canonical structure. In Section 6, we present effective algorithms to derive the stratification by combinatorial rules acting on the integer partitions representing the canonical structure information. The StratiGraph software tool[1] [22,24] was developed to compute and visualize the closure hierarchy graphs.

[1] StratiGraph can be downloaded from

Before we introduce the theory of stratification we recall in Section 2 a canonical form for the system pencils $\mathcal{S}$ (2) and $\mathcal{S}_0$ (3) under feedback-injection equivalence [25,27]. A number of system characteristics are invariant under feedback-injection [6], i.e., they depend directly on and may be derived from the canonical forms. Examples include: invertibility, i.e., existence of the rational matrix function which is the right/left inverse to the transfer function of $\mathcal{S}$; normal rank, i.e., the maximum rank of the transfer function of $\mathcal{S}$ attainable with some s; and the finite and infinite invariant zero structures. Notably, characteristics that are not invariant under feedback-injection are controllability, observability, and the poles of the system. Finally, in Section 7 we illustrate the stratification theory and investigate some of the above invariants by an example.

2 Canonical forms

In this section, we recall the Kronecker canonical form of a general matrix pencil $M - sN$ as well as the canonical form of a system pencil $\mathcal{S}$ under feedback-injection equivalence.

For each $k = 1, 2, \ldots$, define the $k \times k$ matrices

$$J_k(\mu) := \begin{bmatrix} \mu & 1 & & \\ & \mu & \ddots & \\ & & \ddots & 1 \\ & & & \mu \end{bmatrix}, \qquad I_k := \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \tag{4}$$

$$Z_k := \begin{bmatrix} & & & 1 \\ & & \iddots & \\ & 1 & & \\ 1 & & & \end{bmatrix}, \qquad Y_k := \begin{bmatrix} & & & 1 & 0 \\ & & \iddots & \iddots & \\ 1 & 0 & & & \\ 0 & & & & \end{bmatrix}, \tag{5}$$

where $\mu \in \mathbb{C}$, and for each $k = 0, 1, \ldots$, define the $k \times (k+1)$ matrices

$$G_k := \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}, \qquad H_k := \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}. \tag{6}$$

All non-specified entries of $J_k(\mu), I_k, Z_k, Y_k, G_k$, and H_k are zeros.

---

http://www.cs.umu.se/english/research/groups/matrix-computations/stratigraph

2.1 Kronecker canonical form

An $\widehat{m} \times \widehat{n}$ matrix pencil $M - sN$ is called *strictly equivalent* to $\widetilde{M} - s\widetilde{N}$ if there exist non-singular matrices Q and R of conforming sizes such that $Q^{-1}MR = \widetilde{M}$ and $Q^{-1}NR = \widetilde{N}$. The set of matrix pencils strictly equivalent to $M - sN$ forms a manifold in the complex $2\widehat{m}\widehat{n}$ dimensional space. This manifold is the orbit of $M - sN$ under the action of equivalence:

$$\mathcal{O}^{\mathrm{e}}_{M-sN} = \{Q^{-1}(M - sN)R \mid \det(Q) \cdot \det(R) \neq 0\}. \tag{7}$$

It follows that all matrix pencils in this orbit have the same *Kronecker Canonical Form* (KCF) which is stated by the following theorem.

Theorem 1 (Sec. XII.4, [18]) *Each $m \times n$ matrix pencil $M - sN$ is equivalent to a direct sum of pencils of the forms*

$$J_h(\mu) - sI_h, \ \textit{where } \mu \in \mathbb{C}, \quad I_q - sJ_q(0), \quad G_\epsilon - sH_\epsilon, \quad \textit{and} \ \ G_\eta^T - sH_\eta^T,$$

where the blocks are defined in (4) *and* (6)*. This sum is uniquely determined up to permutation of the summands.*

The $h \times h$ Jordan block $J_h(\mu)$, associated with the finite eigenvalue μ, corresponds to a *finite elementary* divisor $(s - \mu)^h$ of degree h. Similarly, $J_q(0)$ is a nilpotent Jordan block and the pencil $I_q - sJ_q(0)$, associated with the infinite eigenvalue, corresponds to an *infinite elementary divisor* $1/s^q$ of degree q. The remaining matrix pencils $G_\epsilon - sH_\epsilon$ and $G_\eta^T - sH_\eta^T$ form the right and left singular parts, respectively, where ϵ corresponds to a *column (right) minimal index* and η to a *row (left) minimal index* of the matrix pencil.

2.2 Canonical form under feedback-injection equivalence

Two system pencils $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ of the form (2) with non-singular E and $\widetilde{E}$ are called *feedback-injection equivalent* if there exist non-singular matrices

$$U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix} \tag{8}$$

of sizes $(n+p) \times (n+p)$ and $(n+m) \times (n+m)$, respectively, such that $U\mathcal{S}V = \widetilde{\mathcal{S}}$, i.e.,

$$\begin{aligned} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix} &= \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix}, \\ \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix} &= \begin{bmatrix} \widetilde{E} & 0 \\ 0 & 0 \end{bmatrix}, \quad (\ U_{11}EV_{11} = \widetilde{E} \ ). \end{aligned} \tag{9}$$

The set of matrix pencils equivalent to $\mathcal{S}$ forms a manifold in the complex $(n+m)(n+p)+n^2$ dimensional space. This manifold is the *orbit of* $\mathcal{S}$ *under feedback-injection equivalence*:

$$\mathcal{O}^{\text{f-i}}_{\mathcal{S}} = \{U\mathcal{S}V \mid U \text{ and } V \text{ are non-singular of the form (8)}\}. \qquad (10)$$

The *block direct sum* $\mathcal{S} \boxplus \mathcal{S}'$ of the pencils

$$\mathcal{S} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} - s\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{S}' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} - s\begin{bmatrix} E' & 0 \\ 0 & 0 \end{bmatrix},$$

is defined as

$$\mathcal{S} \boxplus \mathcal{S}' := \begin{bmatrix} A\oplus A' & B\oplus B' \\ C\oplus C' & D\oplus D' \end{bmatrix} - s\begin{bmatrix} E\oplus E' & 0 \\ 0 & 0 \end{bmatrix},$$

where $\oplus$ is the direct sum of matrices. In order to define the blocking of a pencil $\mathcal{S}$, we write $\mathcal{S}_{pm}$, where $p \times m$ is the size of the D matrix.

The following theorem presents a canonical form of (2) under feedback-injection equivalence.

Theorem 2 ([27]) *A system pencil* $\mathcal{S}$*, defined in* (2)*, is feedback-injection equivalent to a block direct sum of system pencils of the forms*

$$(J_h(\mu) - sI_h)_{00}, \; (Z_q - sY_q)_{11}, \; (G_\epsilon - sH_\epsilon)_{01}, \; \textit{and} \; (G_\eta^T - sH_\eta^T)_{10}, \qquad (11)$$

where the pencils are defined as in (4)–(6) *and their blockings are defined by the corresponding subscripts (*$00, 11, 01,$ *or* 10*). This sum is uniquely determined by* $\mathcal{S}$*, up to permutation of the block direct summands.*

By Theorem 2 each system pencil can be expressed as a block direct sum of the canonical blocks as follows

$$\boxplus_j \boxplus_i (J_{h_i}(\mu_j) - sI_{h_i})_{00} \;\boxplus\; \boxplus_i (Z_{q_i} - sY_{q_i})_{11} \;\boxplus\; \boxplus_i (G_{\epsilon_i} - sH_{\epsilon_i})_{01} \;\boxplus\; \boxplus_i (G^T_{\eta_i} - sH^T_{\eta_i})_{10}. \qquad (12)$$

It is well-known that a generalized finite or infinite eigenvalue of the system pencil $\mathcal{S}$ (associated with $(J_h(\mu) - sI_h)_{00}$ and $(Z_q - sY_q)_{11}$, respectively) is a *finite* or *infinite invariant zero* for the corresponding state-space system (1), e.g., see [1]. If $\mathcal{S}$ is a minimal realization (both controllable and observable) then the finite invariant zeros coincide with the *transmission zeros*. For the finite eigenvalues there is a one-to-one correspondence between the degrees h_i of the elementary divisors and the orders of the invariant zeros. For the infinite eigenvalues the orders differ by one; the orders of the infinite invariant zeros are $q_i - 1$. In other words, the infinite elementary divisors of degree one do not appear as zeros, they actually act as a constant feed-through and are therefore sometimes called non-dynamic variables [28]. Notably, for a single-input single-output system the order of the infinite zero is equal to the relative degree of the corresponding transfer function. For a full description of the different types of zeros and their relations we refer to [1,6].

Any existing $(G_0 - sH_0)_{01}$ or $(G_0^T - sH_0^T)_{10}$ corresponds to a redundant system input or output, respectively. An important invariant under feedback-injection equivalence associated with the remaining singular blocks is invertibility [6,26] of the corresponding transfer function $H(s) = C(sE - A)^{-1}B + D, \det(E) \neq 0$, of the state-space system (1). The system is said to be *left* (or *right*) *invertible* if there exists a rational function $L(s)$ (or $R(s)$) such that $L(s)H(s) = I_m$ (or $H(s)R(s) = I_p$). The system is invertible if it is both right and left invertible. Invertibility of the system can also be studied from the canonical form (11). Assuming that B and C have full rank, a system is left (or right) invertible if it has no column (or row) minimal indices greater than zero, i.e., no $G_\epsilon - sH_\epsilon$ (or $G_\eta^T - sH_\eta^T$) summands where $\{\epsilon, \eta\} > 0$. It follows that an invertible system must have the same number of inputs m as outputs p, but the opposite does not need to hold.

The KCF of a system pencil $\mathcal{S}$ in (2), given in the canonical form (12), can be obtained by substituting the block direct sum of the pencils in (12) with the direct sum of the corresponding Kronecker canonical pencils obtained as follows. Every pencil $(Z_q - sY_q)_{11}$ is replaced by a strictly equivalent pencil $I_q - sJ_q(0)$. For the other types of summands it is enough just to ignore (drop) the blocking. In other words, the transformation from (12) to the KCF can be obtained by a strict equivalence transformation using permutation matrices.

Of course, all system pencils $\mathcal{S}$ can be transformed into KCF but not all $(n+m) \times (n+p)$ general matrix pencils can be transformed into a system pencil in the canonical form (12) with n states, m inputs, and p outputs. Nevertheless, there is only one restriction: the dimensions of the three zero-blocks in the s-matrix of (2) must be matched. This leads to the following theorem.

Theorem 3 *There exists a system pencil of the form* (2) *that has n states, p inputs, and m outputs, with the set of Kronecker invariants $\{\eta_i\}$ (row minimal indices), $\{\epsilon_i\}$ (column minimal indices), $\{q_i\}$ (associated with the infinite elementary divisors), and $\{h_i^j\}$ (associated with the finite elementary divisors of the eigenvalue μ_j) if and only if*

$$\sum_i \eta_i + \sum_i \epsilon_i + \sum_j \sum_i h_i^j + \sum_i (q_i - 1) = n, \tag{13}$$

$$\sum_i \#\{\eta_i\} + \sum_i \#\{q_i\} = m, \tag{14}$$

$$\sum_i \#\{\epsilon_i\} + \sum_i \#\{q_i\} = p, \tag{15}$$

where $\#\{x_i\}$ denotes the number of elements in the set $\{x_i\}$. Some of the sets of invariants may be empty.

The following lemma states that feedback-injection equivalence and strict equivalence are interchangeable notions for the system pencils.

Lemma 4 (see also [7,20]) *Two system pencils of the form* (2) *are feedback-injection equivalent if and only if they are strictly equivalent.*

PROOF. The sufficiency is obvious. Let us show the necessity by considering the s-matrix of the system pencils:

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} E_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Performing the matrix multiplication we obtain

$$\begin{bmatrix} R_{11}E_1S_{11} & R_{11}E_1S_{12} \\ R_{21}E_1S_{11} & R_{21}E_1S_{12} \end{bmatrix} = \begin{bmatrix} E_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since (by assumption) both $n \times n$ matrices E_1 and E_2 have full rank and $R_{11}E_1S_{11} = E_2$, we have that R_{11} and S_{11} must also have rank n. Therefore, the remaining equations imply that $R_{21} = 0$ and $S_{12} = 0$. $\square$

3 Versal deformations of matrix pencils

While studying matrices depending on parameters, V.I. Arnold introduced the concept of a *(mini)versal deformation* of a matrix with respect to similarity [2] (see also [4, Ch. 30B]). Later versal deformations have been obtained fro matrix pencils [15], as well as for matrix pencils with symmetries [10,13]. Recall that a *deformation* of an $(n+m)\times(n+p)$ matrix pencil $\mathcal{P}$ is a holomorphic mapping $\mathcal{P}(\vec{\varepsilon})$, where $\vec{\varepsilon} := (\varepsilon_1, \dots, \varepsilon_k)$, from a neighbourhood $\Omega \subset \mathbb{C}^k$ of $\vec{0} = (0, \dots, 0)$ to the space of $(n+m) \times (n+p)$ matrix pencils such that $\mathcal{P}(\vec{0}) = \mathcal{P}$. A deformation $\mathcal{P}(\varepsilon_1, \dots, \varepsilon_k)$ of a matrix pencil $\mathcal{P}$ is called *versal* if for every deformation $\mathcal{Q}(\delta_1, \dots, \delta_l)$ of $\mathcal{P}$ we have

$$\mathcal{Q}(\delta_1, \dots, \delta_l) = \mathcal{I}_1(\delta_1, \dots, \delta_l)\mathcal{P}(\varphi_1(\vec{\delta}), \dots, \varphi_k(\vec{\delta}))\mathcal{I}_2(\delta_1, \dots, \delta_l),$$

where $\mathcal{I}_1(\delta_1, \dots, \delta_l)$ and $\mathcal{I}_2(\delta_1, \dots, \delta_l)$ are deformations of the identity matrices, and all $\varphi_i(\vec{\delta})$ are convergent in a neighborhood of $\vec{0}$ power series such that $\varphi_i(\vec{0}) = 0$. A versal deformation $\mathcal{P}(\varepsilon_1, \dots, \varepsilon_k)$ of $\mathcal{P}$ is called *miniversal* if there exists no versal deformation having less than k parameters.

In this section, we will consider the system pencil $\mathcal{S}$ (2) as a general matrix pencil under strict equivalence $\mathcal{S} \mapsto U\mathcal{S}V$, where

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \quad V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}, \quad \det(U) \cdot \det(V) \neq 0, \tag{16}$$

and investigate all matrix pencils in a neighbourhood of $\mathcal{S}$, i.e.,

$$\mathcal{S} + \mathcal{W} := \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} W_1 & W_3 \\ W_2 & W_4 \end{bmatrix} - s\left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} W_5 & W_7 \\ W_6 & W_8 \end{bmatrix}\right). \tag{17}$$

In particular, we allow perturbations of the zero blocks in the s-matrix of $\mathcal{S}$ and thus the form (2) of $\mathcal{S}$ is not required to be preserved, e.g., the rank of the s-matrix may change. Our goal is to find a matrix pencil $\mathcal{S}(\mathcal{W})$ to which all $(n+m) \times (n+p)$ matrix pencils $\mathcal{S} + \mathcal{W}$ that are close to a given $\mathcal{S}$, can be reduced by

$$\mathcal{S} + \mathcal{W} \mapsto U(\mathcal{W})(\mathcal{S} + \mathcal{W})V(\mathcal{W}) =: \mathcal{S}(\mathcal{W}), \tag{18}$$

in which $U(\mathcal{W})$ and $V(\mathcal{W})$ are holomorphic at 0 (i.e., its entries are power series in the entries of $\mathcal{W}$ that are convergent in a neighborhood of 0), $U(0)$ and $V(0)$ are non-singular matrices. By choosing $U(0)$ and $V(0)$ to be identities and (18), we have $\mathcal{S}(0)$ equal to $\mathcal{S}$. Define a matrix pencil $\mathcal{D}(\mathcal{W})$ from

$$\mathcal{S} + \mathcal{D}(\mathcal{W}) = U(\mathcal{W})(\mathcal{S} + \mathcal{W})V(\mathcal{W}). \tag{19}$$

Therefore $\mathcal{D}(\mathcal{W})$ is holomorphic at 0 and $\mathcal{D}(0) = 0$. We have that $\mathcal{S} + \mathcal{D}(\mathcal{W})$ is a versal deformation of $\mathcal{S}$, see also [10,13].

Following the notation in [13] where the big "+" denotes the entrywise sum of matrices, define $\mathcal{D}(\mathbb{C})$ to be a space of all matrix pencils of the form $\mathcal{D}(\mathcal{W})$, i.e., each nonzero entry (i,j) of $\mathcal{D}(\mathcal{W})$ is replaced by a complex number:

$$\mathcal{D}(\mathbb{C}) := \left(\mathop{\Large +}_{(i,j)\in\mathcal{J}_1(\mathcal{D})} \mathbb{C}F^{(ij)}\right) - s\left(\mathop{\Large +}_{(i,j)\in\mathcal{J}_2(\mathcal{D})} \mathbb{C}F^{(ij)}\right),$$

where

$$\mathcal{J}_1(\mathcal{D}), \mathcal{J}_2(\mathcal{D}) \subseteq \{1, \ldots, n+m\} \times \{1, \ldots, n+p\} \tag{20}$$

are the sets of indices of the nonzero entries in the constant term and the s-term, respectively, of the pencil $\mathcal{D}(\mathcal{W})$, and $F^{(ij)}$ is a matrix whose (i,j)-th entry is 1 and all other entries are 0. For brevity we introduce $\mathbb{C}_{n,m,p} := \mathbb{C}^{(n+m)\times(n+p)} \times \mathbb{C}^{(n+m)\times(n+p)}$.

Lemma 5 *Let $\mathcal{S}$ be of the form* (2) *and $\mathcal{D}(\mathcal{W}) \in \mathbb{C}_{n,m,p}$. The deformation $\mathcal{S} + \mathcal{D}(\mathcal{W})$ is versal if and only if the vector space $\mathbb{C}_{n,m,p}$ decomposes into the sum $T_{\mathcal{S}} + \mathcal{D}(\mathbb{C})$, where $T_{\mathcal{S}}$ is the tangent space to the strict equivalence orbit of $\mathcal{S}$ at the point $\mathcal{S}$.*

PROOF. In a small neighbourhood of the point $\mathcal{S}$ only linear deformations matter and the curvature of the orbit becomes unimportant (see [3, Sec. 1.6] or [2,15]). This allows us to "associate" the orbit of $\mathcal{S}$ with the tangent space to the orbit of $\mathcal{S}$ at the point $\mathcal{S}$. Therefore a versal deformation of $\mathcal{S}$ is transversal to $T_{\mathcal{S}}$ (two subspaces of a vector space are called *transversal* if their sum is equal to the whole space, [4, Ch. 29]). □

The following lemma presents a versal deformation of the matrix pencil (2). It states that we may not perturb the off-diagonal blocks of the s-term, i.e., without loss of generality we may assume that $W_6 = 0$ and $W_7 = 0$ in (17).

Lemma 6 *Let $\mathcal{S}$ be a matrix pencil of the form* (2)*. Its versal deformation can be taken in the form*

$$\mathcal{S} + \mathcal{D}(\mathcal{W}) := \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} W_1 & W_3 \\ W_2 & W_4 \end{bmatrix} - s\left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} W_5 & 0 \\ 0 & W_8 \end{bmatrix}\right) \tag{21}$$

in which $W_i, i \in \{1,2,3,4,5,8\}$ are matrices with arbitrarily small entries (all entries are independent from each other).

PROOF. The tangent space to the strict equivalence orbit of $\mathcal{S}$ can be represented as follows

$$\begin{aligned} T_{\mathcal{S}} &:= \left\{\left(\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}\right) - s\left(\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}\right)\right\} \\ &= \left\{\begin{bmatrix} U_{11}A + AV_{11} + U_{12}C + BV_{21} & U_{11}B + BV_{22} + AV_{12} + U_{12}D \\ U_{22}C + CV_{11} + U_{21}A + DV_{21} & U_{22}D + DV_{22} + CV_{12} + U_{21}B \end{bmatrix} - s\begin{bmatrix} U_{11}E + EV_{11} & EV_{12} \\ U_{21}E & 0 \end{bmatrix}\right\}, \end{aligned} \tag{22}$$

where $U = [U_{ij}]$ and $V = [V_{ij}]$ are any block matrices of conforming sizes. Since E is non-singular the matrices EV_{12} and $U_{21}E$ can be arbitrary. Thus the entries of the corresponding blocks in the matrix pencils from $\mathcal{D}(\mathbb{C})$ can be taken as zero elements and (21) holds by Lemma 5. □

Note that we do not aim for the miniversal deformation (see [13] for more details) but we want the deformation to be "good enough" for our stratification purposes.

Actually, the dimension of the tangent space $T_{\mathcal{S}}$ is equal to the dimension of the orbit of $\mathcal{S}$, while the minimal dimension of a transversal to $T_{\mathcal{S}}$ is the codimension of the orbit of $\mathcal{S}$. Computing codimensions of the orbits is another question of interest since the codimensions give a coarse stratification: only orbits with higher codimensions may be in the closure of a given orbit. The codimensions for system pencils are presented in [19]; below we state our simplified (but equivalent) formula.

Theorem 7 *Let $\mathcal{S}$ be a matrix pencil of the form* (2). *Then*

$$\operatorname{codim} \mathcal{O}_{\mathcal{S}}^{\text{f-i}} = \operatorname{codim} \mathcal{O}_{\mathcal{S}}^{\text{e}} - mp.$$

PROOF. Note that $\mathcal{S}$ under strict equivalence is considered as an element of the $2(n+m)(n+p)$ dimensional space and $\mathcal{S}$ under feedback-injection equivalence transformations (9) is considered as an element of the $(n+m)(n+p)+n^2$ dimensional space.

Since E in (22) is non-singular, any transversal space to (22) differs from the corresponding transversal space to the tangent space of $\mathcal{S}$ considered as a system pencil only by the $(2,2)$-block of the s-term. The $(2,2)$-block of the s-term has mp elements. □

4 Stratification of system pencils

We are now ready to investigate canonical structure transitions of a system pencil $\mathcal{S}$ under small perturbations $\mathcal{W}'$, i.e., we study which canonical structure information the system pencil

$$\mathcal{S} + \mathcal{W}' = \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} W_1' & W_3' \\ W_2' & W_4' \end{bmatrix} - s\left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} W_5' & 0 \\ 0 & 0 \end{bmatrix}\right) \tag{23}$$

(which represents all system pencils that are close to $\mathcal{S}$) may have. Note that in the following we only perturb the matrices A, B, C, D, and E, coming from the system (1), but not the zeros in the s-term of $\mathcal{S}$, i.e., we preserve the structure of the system pencil.

In the proof we use the following well-known lemma.

Lemma 8 (Sec. 0.4.6, [21]) *Let* $X \in \mathbb{C}^{m\times n}$, $P \in \mathbb{C}^{m\times m}$, $\operatorname{rank}(P) = m$, *and* $Q \in \mathbb{C}^{n\times n}$, $\operatorname{rank}(Q) = n$ *then* $\operatorname{rank}(X) = \operatorname{rank}(PXQ)$.

Theorem 9 *Let $\mathcal{S}$ and $\mathcal{Q}$ be two pencils of the form* (2) *(with the same block sizes). Then there exists a $\mathcal{W}'$ such that $\mathcal{S} + \mathcal{W}'$* (23) *is feedback-injection equivalent to $\mathcal{Q}$ if and only if there exists a $\mathcal{W}$ such that $\mathcal{S} + \mathcal{W}$* (17) *is strictly equivalent to $\mathcal{Q}$.*

PROOF. The feedback-injection obviously implies the strict equivalence. Now we prove the other direction. Both $\mathcal{S}$ and $\mathcal{Q}$ are system pencils that have the same block-structure with non-singular E matrices. By the assumption that there is a perturbation of $\mathcal{S}$ (as a matrix pencil) and two non-singular matrices U and V such that

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} W_1 & W_3 \\ W_2 & W_4 \end{bmatrix} - s \left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} W_5 & W_7 \\ W_6 & W_8 \end{bmatrix} \right) \right) \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \mathcal{Q},$$

there exists a perturbation in the form of a versal deformation strictly equivalent to $\mathcal{Q}$. So by Lemma 6 we may assume that $W_6 = 0$ and $W_7 = 0$. Furthermore, if $W_8 \neq 0$ then the rank of the second matrix will increase by Lemma 8. Thus there must exist a perturbation

$$\mathcal{W}' := \begin{bmatrix} W_1' & W_3' \\ W_2' & W_4' \end{bmatrix} - s \begin{bmatrix} W_5' & 0 \\ 0 & 0 \end{bmatrix},$$

such that $\mathcal{S} + \mathcal{W}'$ and $\mathcal{Q}$ are strictly equivalent and thus they are feedback-injection equivalent by Lemma 4. $\square$

By Theorem 9 we have that $U^{-1}\mathcal{Q}V^{-1} - \mathcal{S} = \mathcal{W}$ and $(U')^{-1}\mathcal{Q}(V')^{-1} - \mathcal{S} = \mathcal{W}'$, where U, V and U', V' represent strict and feedback-injection equivalences, respectively. Since the entries of $\mathcal{W}$ and $\mathcal{W}'$ are arbitrarily small we have the following corollary for sequences of matrices with entry-wise convergence.

Corollary 10 *Let $\mathcal{S}$ and $\mathcal{Q}$ be two pencils of the form* (2). *There exists a sequence of non-singular matrices*

$$\left\{ U^{(k)} = \begin{bmatrix} U_{11}^{(k)} & U_{12}^{(k)} \\ U_{21}^{(k)} & U_{22}^{(k)} \end{bmatrix}, \quad V^{(k)} = \begin{bmatrix} V_{11}^{(k)} & V_{12}^{(k)} \\ V_{21}^{(k)} & V_{22}^{(k)} \end{bmatrix} \right\}, \quad \text{such that} \quad U^{(k)}\mathcal{Q}V^{(k)} \to \mathcal{S}$$

if and only if there exists a sequence of nonsingular matrices

$$\left\{ U'^{(k)} = \begin{bmatrix} U_{11}'^{(k)} & U_{12}'^{(k)} \\ 0 & U_{22}'^{(k)} \end{bmatrix}, \; V'^{(k)} = \begin{bmatrix} V_{11}'^{(k)} & 0 \\ V_{21}'^{(k)} & V_{22}'^{(k)} \end{bmatrix} \right\}, \; \text{such that } U'^{(k)}\mathcal{Q}V'^{(k)} \to \mathcal{S}.$$

Below we reformulate Theorem 9 in the form convenient for the proofs in Section 6.1, where the closure of a set X in the Euclidean topology is denoted by $\overline{X}$.

Corollary 11 *Let $\mathcal{S}$ and $\mathcal{Q}$ be two pencils of the form* (2). *Then*

$$\overline{\mathcal{O}^{\text{f-i}}_{\mathcal{Q}}} \supset \mathcal{O}^{\text{f-i}}_{\mathcal{S}} \quad \text{if and only if} \quad \overline{\mathcal{O}^{\text{e}}_{\mathcal{Q}}} \supset \mathcal{O}^{\text{e}}_{\mathcal{S}}.$$

From the closure relations between the orbits of system pencils a hierarchical graph of orbits can be formed; known as a *closure hierarchy graph* or a *stratification.* Each node (vertex) in the graph represents an orbit and each edge represents a cover/closure relation. In the graph, there is an upward path from a node representing $\mathcal{S}$ to a node representing $\mathcal{Q}$ if and only if $\mathcal{S}$ can be transformed by an arbitrarily small perturbation to a system pencil whose canonical form is the one of $\mathcal{Q}$. As a result, we get qualitative information about the nearby system pencils and associated canonical forms.

Theorem 9 reveals the connections between the behaviour of general matrix pencils and system pencils under small perturbations. More precisely, Theorem 9 shows that the stratification of system pencils (2) can be extracted from the stratification of $(n+m)\times(n+p)$ matrix pencils under strict equivalence [15,16]: by Theorem 3 the nodes that correspond to system pencils are extracted and by Theorem 9 an edge between two extracted nodes is placed if and only if there is a path directed upward (may be passing through other nodes) between the corresponding nodes in the $(n+m)\times(n+p)$ matrix pencil stratification. This method essentially requires checking all the possible pairs of nodes. Another more efficient method which allows finding the neighbouring structures explicitly is presented in Section 6. An illustrative example of a subgraph of a stratification graph is presented in Section 7.

5 Stratification of system pencils without feedforward

We now consider the special case of a strictly proper system (1) with no direct feedforward term, i.e., $D \equiv 0$, with the associated system pencil $\mathcal{S}_0$ in (3). The canonical form of $\mathcal{S}_0$ under feedback-injection equivalence is similar to the one presented in Theorem 2 except that the blocks $(Z_1 - sY_1)_{11}$ corresponding to the infinite elementary divisors of degree one, cannot appear, which leads to the following corollary of Theorem 3.

Corollary 12 *There exists a system pencil of the form* (3) *with the set of Kronecker invariants* $\{l_i\}, \{r_i\}, \{q_i\}$, *and* $\{h_i^j\}$, *where* $q_i \geqslant 2$ *for all* i, *if and only if* (13)–(15) *hold. Some of the sets of invariants may be empty.*

From Lemma 6 we have the following corollary for versal deformations of $\mathcal{S}_0$.

Corollary 13 *Let* $\mathcal{S}_0$ *be a matrix pencil of the form* (3). *Then* $\mathcal{S}_0 + \mathcal{D}(\mathcal{W})$ *is a versal deformation of* $\mathcal{S}_0$ *in the form* (21) *with* $D = 0$.

The pencil

$$\mathcal{S}_0 + \mathcal{W}_0 := \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + \begin{bmatrix} W_1 & W_3 \\ W_2 & 0 \end{bmatrix} - s\left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} W_5 & 0 \\ 0 & 0 \end{bmatrix}\right) \tag{24}$$

represents all system pencils that are close to $\mathcal{S}_0$ (all W_i have arbitrarily small entries). The stratification of system pencils $\mathcal{S}_0$ follows now from the following theorem which is an analogue of Theorem 9.

Theorem 14 *Let $\mathcal{S}_0$ and $\mathcal{Q}_0$ be two pencils of the form* (3)*. Then there exists a $\mathcal{W}_0$ such that $\mathcal{S}_0 + \mathcal{W}_0$* (24) *is feedback-injection equivalent to $\mathcal{Q}_0$ if and only if there exists a $\mathcal{W}$ such that $\mathcal{S}_0 + \mathcal{W}$* (17) *is strictly equivalent to $\mathcal{Q}_0$.*

PROOF. Like in Theorem 9 the forward direction is obvious. Using Corollary 13 instead of Lemma 6 we can repeat the proof of Theorem 9 obtaining

$$\begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} \left(\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + \begin{bmatrix} W_1 & W_3 \\ W_2 & W_4 \end{bmatrix} - s \left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} W_5 & 0 \\ 0 & 0 \end{bmatrix} \right) \right) \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix} = \mathcal{Q}_0$$

If we perturb any entry of the $(2,2)$-block in the constant matrix of the system pencil $\mathcal{S}_0$, i.e. we take $W_4 \neq 0$, then the corresponding block of $\mathcal{Q}_0$ will be equal to $U_{22}W_4V_{22} \neq 0$ (since both U_{22} and V_{22} are nonsingular). Thus W_4 must be equal to zero, which proves the theorem. $\square$

The above presented theory justifies the extraction of the stratification of system pencils (3) from the stratification of $(n+m) \times (n+p)$ matrix pencils under strict equivalence [15,16] in the same way as it was done for system pencils (2).

6 Neighbouring structures in the stratification

A sequence of integers $\mathcal{N} = (n_1, n_2, n_3, \dots)$ such that $n_1 + n_2 + n_3 + \dots = n$ and $n_1 \geqslant n_2 \geqslant \dots \geqslant 0$ is called an *integer partition* of n (for more details and references see [15]). For any $a \in \mathbb{Z}_{\geqslant 0}$ we define $\mathcal{N} + a$ as follows $(n_1 + a, n_2 + a, n_3 + a, \dots)$. The set of all integer partitions form a poset (even a lattice) with respect to the following order $\mathcal{N} \geq \mathcal{M}$ if and only if $n_1 + n_2 + \dots + n_i \geqslant m_1 + m_2 + \dots + m_i$, for $i \geqslant 1$. When $\mathcal{N} \geq \mathcal{M}$ and $\mathcal{N} \neq \mathcal{M}$ then $\mathcal{N} > \mathcal{M}$. If $\mathcal{N}, \mathcal{M}$ and $\mathcal{K}$ are integer partitions of the same integer n and there does not exist any $\mathcal{K}$ such that $\mathcal{N} > \mathcal{K} > \mathcal{M}$ where $\mathcal{N} > \mathcal{M}$, then $\mathcal{N}$ *covers* $\mathcal{M}$.

An integer partition $\mathcal{N} = (n_1, n_2, n_3, \dots)$ can also be represented by n piles of coins, where the first pile has n_1 coins, the second n_2 coins and so on. An integer partition $\mathcal{N}$ covers $\mathcal{M}$ if $\mathcal{M}$ can be obtained from $\mathcal{N}$ by moving one coin one column rightwards or one row downwards, and keep $\mathcal{N}$ monotonically non-increasing. Or equivalently, an integer partition $\mathcal{N}$ is *covered by* $\mathcal{K}$ if $\mathcal{K}$ can be obtained from $\mathcal{K}$ by moving one coin one column leftwards or one row upwards, and keep $\mathcal{N}$ monotonically non-increasing. These two types of coin moves are defined in [16] and called *minimum rightward* and *minimum leftward coin moves*, respectively, see Figure 1.

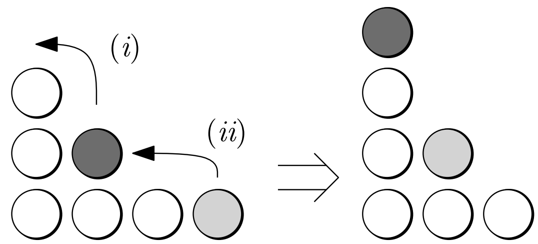

Fig. 1. To the partition on the left we apply two minimal leftward coin moves: (i) is a move of a dark-grey coin one column leftward and (ii) is a move of a light-grey coin one row upward. Note that monotonicity must be preserved. The resulting partition is on the right.

With every system pencil $\mathcal{S}$ (with eigenvalues $\mu_i \in \mathbb{C}$), following [16], we associate the set of *structure integer partitions* $\mathcal{R}(\mathcal{S}), \mathcal{L}(\mathcal{S}), \mathcal{Z}(\mathcal{S})$, and $\{\mathcal{J}_{\mu_j}(\mathcal{S}) : j = 1, \dots, d\}$, where d is the number of distinct eigenvalues of $\mathcal{S}$. These partitions, known as Weyr characteristics are created as follows:

- For each distinct μ_j, $\mathcal{J}_{\mu_j}(\mathcal{S}) = (h_1^{\mu_j}, h_2^{\mu_j}, \dots)$: the k^{th} position is the number of Jordan blocks of size greater than or equal to k; the position numeration starting from 1.
- $\mathcal{Z}(\mathcal{S}) = (z_1, z_2, \dots)$: the k^{th} position is the number of $Z - sY$ blocks of size greater than or equal to k; the position numeration starting from 1.
- $\mathcal{R}(\mathcal{S}) = (r_0, r_1, \dots)$: the k^{th} position is the number of $G - sH$ blocks of size greater than or equal to $k \times (k+1)$; the position numeration starting from 0.
- $\mathcal{L}(\mathcal{S}) = (l_0, l_1, \dots)$: the k^{th} position is the number of $G^T - sH^T$ blocks of size greater than or equal to $(k+1) \times k$; the position numeration starting from 0.

6.1 Neighbouring orbits in the stratification

By expressing the Kronecker indices as structure integer partitions we can express the cover relations between two orbits by utilizing minimal coin moves and combinatorial rules on the integer partitions.

We say that the feedback-injection orbit $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_1}$ *covers* $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_2}$ if and only if $\overline{\mathcal{O}^{\text{f-i}}_{\mathcal{S}_1}} \supset \mathcal{O}^{\text{f-i}}_{\mathcal{S}_2}$ and there exists no orbit $\mathcal{O}^{\text{f-i}}_{\mathcal{Q}}$ such that $\overline{\mathcal{O}^{\text{f-i}}_{\mathcal{S}_1}} \supset \mathcal{O}^{\text{f-i}}_{\mathcal{Q}}$ and $\overline{\mathcal{O}^{\text{f-i}}_{\mathcal{Q}}} \supset \mathcal{O}^{\text{f-i}}_{\mathcal{S}_2}$; or equivalently, if and only if there is an (upward) edge from $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_2}$ to $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_1}$ in the orbit stratification graph.

The main idea of the proofs of Theorems 15 and 17 (as well as Theorems 18 and 19) is: from the corresponding sets of rules for general matrix pencils we exclude the rules that do not preserve the block form of the system pencil. By preserving the system pencil structure we mean that if the rules are applied to a system pencil that has n states, p inputs, and m outputs then the resulting system pencils also have n states, p inputs, and m outputs.

Theorem 15 *The feedback-injection orbit* $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_1}$ covers $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_2}$ *if and only if the structure integer partitions of* $\mathcal{S}_2$ *can be obtained by applying one of the rules (1)–(6) to the structure integer partitions of* $\mathcal{S}_1$*:*

(1) Minimum rightward coin move in $\mathcal{R}$ *(or* $\mathcal{L}$*).*
(2) If the rightmost column in $\mathcal{R}$ *(or* $\mathcal{L}$*) is one single coin, move that coin to a new rightmost column of some* $\mathcal{J}_{\mu_i}$*, which may be empty initially.*
(3) If the rightmost column in $\mathcal{R}$ *(or* $\mathcal{L}$*) is one single coin and* $\mathcal{Z}$ *is non-empty, move that coin to a new rightmost column of* $\mathcal{Z}$*.*
(4) Minimum leftward coin move in any $\mathcal{J}_{\mu_i}$*.*
(5) Minimum leftward coin move in $\mathcal{Z}$ *but no moves to the first (= leftmost) column are allowed.*
(6) If $\mathcal{Z}$ *is non-empty: Let* k *denote the total number of coins in all of the longest (= lowest) rows from all of the* $\mathcal{J}_{\mu_i}$ *and* $\mathcal{Z}$ *together. Remove these* k *coins, add one more coin to the set, and distribute* $k+1$ *coins to* $r_p, p = 0, \dots, t$ *and* $l_q, q = 0, \dots, k-t-1$ *such that at least all nonzero columns of* $\mathcal{R}$ *and* $\mathcal{L}$ *are given coins.*

Rules (1)–(3) may not make coin moves that affect r_0 *(or* l_0*). Rule (6) cannot be applied if the total number of nonzero columns in* $\mathcal{R}$ *and* $\mathcal{L}$ *is greater than* $k+1$*, i.e., if the rule can be applied, at least one coin must be assigned to each column of* $\mathcal{R}$ *and* $\mathcal{L}$*.*

PROOF. The rules in this theorem are restricted forms of the rules for matrix pencils in [16, Theorem 3.2] which preserve the system pencil block form (2). For a system pencil preservation of the block form means that the equalities (13)–(15) in Theorem 3 holds, where (14) and (15) imply that the number of blocks associated with the infinite eigenvalue (z_0 in $\mathcal{Z}$) contributes to m and p but the number of blocks associated with the finite eigenvalues ($h_1^{\mu_j}$ in $\mathcal{J}_{\mu_j}$) do not. Therefore $\mathcal{Z}$ must be treated differently from $\mathcal{J}_{\mu_j}$ (contrary to the matrix pencils case), i.e., $\mathcal{Z}$ must be non-empty in rules (3) and (6) and no moves to the first column are allowed in rule (5). Summing up, let us explicitly point out that, comparing to the rules for matrix pencils in [16, Theorem 3.2], (3) and (5) are new, as well as the condition that $\mathcal{Z}$ must be non-empty in (6), and the other rules are the same.

Now we prove that the set of rules (1)–(6) that preserves the system pencil's block form (2) are necessary and sufficient for a feedback-injection orbit to cover another feedback-injection orbit.

Sufficiency: We have that $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_1}$ and $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_2}$ are two orbits of system pencils and $\mathcal{S}_2$ can be obtained by applying one of the rules (1)–(6) to the integer partitions of $\mathcal{S}_1$. The rules (1)–(6) are more restrictive than the corresponding rules for matrix pencils thus if we consider the strict equivalence orbits of $\mathcal{S}_1$ and $\mathcal{S}_2$

(i.e., $\mathcal{O}^{\mathrm{e}}_{\mathcal{S}_1}$ and $\mathcal{O}^{\mathrm{e}}_{\mathcal{S}_2}$) we get that the integer partitions of $\mathcal{S}_2$ can be obtained by applying one of the rules for matrix pencils to the integer partitions of $\mathcal{S}_1$. Therefore, $\mathcal{O}^{\mathrm{e}}_{\mathcal{S}_1}$ covers $\mathcal{O}^{\mathrm{e}}_{\mathcal{S}_2}$. Due to the fact that the set of system pencil orbits under the feedback-injection equivalence is a subset of the set of matrix pencil orbits under the strict equivalence (see Theorem 2) no new nodes (orbits) can appear in-between $\mathcal{S}_1$ and $\mathcal{S}_2$ in the system stratification, thus $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_1}$ covers $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_2}$ for the system pencils too.

Necessity: We have that $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_1}$ and $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_2}$ are two orbits of system pencils and $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_1}$ covers $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_2}$. Therefore $\overline{\mathcal{O}^{\mathrm{e}}_{\mathcal{S}_1}} \supset \mathcal{O}^{\mathrm{e}}_{\mathcal{S}_2}$ by Corollary 11. If also $\mathcal{O}^{\mathrm{e}}_{\mathcal{S}_1}$ covers $\mathcal{O}^{\mathrm{e}}_{\mathcal{S}_2}$ then one of the rules for matrix pencils can be applied and since both $\mathcal{S}_1$ and $\mathcal{S}_2$ are system pencils then one of the rules (1)–(6) above can be applied too. If $\mathcal{O}^{\mathrm{e}}_{\mathcal{S}_1}$ does not cover $\mathcal{O}^{\mathrm{e}}_{\mathcal{S}_2}$, by Theorem 9 we still have that $\overline{\mathcal{O}^{\mathrm{e}}_{\mathcal{S}_1}}$ contains $\mathcal{O}^{\mathrm{e}}_{\mathcal{S}_2}$. Therefore we can obtain $\mathcal{S}_1$ from $\mathcal{S}_2$ by applying a sequence of coin moves. If the first move in this sequence is one of the moves from the list above (rules (1)–(6)) we have a contradiction to the fact that $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_1}$ covers $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_2}$. On the other side, if the first move is any of the moves possible for the matrix pencils but not allowed for the system pencils then they will increase m or p, or both and no other moves can decrease them back. Thus the case when $\mathcal{O}^{\mathrm{e}}_{\mathcal{S}_1}$ does not covers $\mathcal{O}^{\mathrm{e}}_{\mathcal{S}_2}$ is impossible. □

An *induced subgraph* Γ' of a graph Γ is a subset of the vertices of a graph Γ together with any edges whose endpoints are both in this subset.

Corollary 16 *Let $\mathcal{S}$ be a system pencil of the form* (2)*. The stratification graph of $\mathcal{S}$ is an induced subgraph of the stratification graph of $\mathcal{S}$ considered as a general matrix pencil.*

PROOF. The proof of Theorem 15 shows that for two system pencils (under feedback-injection equivalence) the cover relation holds if and only if it holds for them treated as matrix pencils (under strict equivalence). □

6.2 Neighbouring bundles in the stratification

The bundle stratifications were derived for general matrix pencils [16], skew-symmetric matrix pencils and polynomials [8,12], controllability and observability pairs [17], and linearizations of polynomial matrices [11,23]. Here we present stratification of system pencil bundles. For the system pencil case a *bundle* $\mathcal{B}^{\text{f-i}}_{\mathcal{S}}$ is a union of system pencil orbits with the same singular and Z blocks and the same Jordan blocks except that the distinct finite eigenvalues may be different. This definition of bundle is analogous to the one for matrix pencils under strict equivalence [16], except that orbits associated with the infinite eigenvalue (Z blocks) are no longer in the same bundle as the finite eigenvalues but form an independent bundle (see more about defining bundles for matrix pencils in [9]). In addition, the codimension of $\mathcal{B}^{\text{f-i}}_{\mathcal{S}}$ is

$$\text{codim } \mathcal{B}^{\text{f-i}}_{\mathcal{S}} = \text{codim } \mathcal{O}^{\text{f-i}}_{\mathcal{S}} - \#\{\text{distinct finite eigenvalues of } \mathcal{S}\}.$$

Notably, in the orbit stratification the eigenvalues may appear and disappear but they are fixed (can not change). Contrary, in the bundle stratification the eigenvalues may coalesce or split apart. This definition leads to the following set of rules for determining the closest neighbours in the bundle stratification graphs, in particular, the rules (7) and (8) in Theorem 17 correspond to the eigenvalues coalescing and the rules (7) and (8) in Theorem 19 correspond to the eigenvalues splitting apart.

We number the rules such that each rule has the same number as in the corresponding orbit case in Theorem 15.

Theorem 17 $\mathcal{B}^{\text{f-i}}_{\mathcal{S}_1}$ covers $\mathcal{B}^{\text{f-i}}_{\mathcal{S}_2}$ *if and only if* $\mathcal{S}_2$ *can be obtained by applying one of the rules (1)–(6) to the structure integer partitions of* $\mathcal{S}_1$*:*

(1) Minimum rightward coin move in $\mathcal{R}$ *(or* $\mathcal{L}$*).*
(2) If the rightmost column in $\mathcal{R}$ *(or* $\mathcal{L}$*) is one single coin, create with that coin a new partition* $\mathcal{J}_{\mu_i}$ *for some new eigenvalue* μ_i*.*
(3) Not applicable.
(4) Minimum leftward coin move in any $\mathcal{J}_{\mu_i}$*.*
(5) Minimum leftward coin move in $\mathcal{Z}$ *but no moves to the first column are allowed.*
(6) If $\mathcal{Z}$ *is non-empty and either there are no finite eigenvalues or at least two Jordan blocks for each finite eigenvalue: Let* k *denote the total number of coins in all of the longest (= lowest) rows from all of the* $\mathcal{J}_{\mu_i}$ *and* $\mathcal{Z}$ *together. Remove these* k *coins, add one more coin to the set, and distribute* $k+1$ *coins to* $r_p, p = 0, \ldots, t$ *and* $l_q, q = 0, \ldots, k-t-1$ *such that at least all nonzero columns of* $\mathcal{R}$ *and* $\mathcal{L}$ *are given coins.*
(7) Join two partitions corresponding to two different finite eigenvalues into one partition.
(8) If $\mathcal{Z}$ *is non-empty join it with a partition that corresponds to a finite eigenvalue and write the result as a partition* $\mathcal{Z}$*, such that the number of blocks corresponding to infinite eigenvalue, i.e.,* z_0*, remains the same.*

Rules (1)–(2) may not make coin moves that affect r_0 *(or* l_0*).*

PROOF. The proof relies on [16, Theorem 3.3] and essentially repeats the proof of Theorem 15. One issue that appears here is that some bundles of matrix pencils split into few bundles for system pencils: corresponding to finite and infinite eigenvalues.

Remark 6.1 *In [7] the points of continuity of matrix quadruples were obtained. Those points correspond to the most generic structures in the bundle stratifications (the topmost nodes in the graphs) presented in this paper.*

Remark 6.2 *Using Corollaries 12 and 13 we may state the analogous rules for the system pencils without feedforward. The absence of the blocks* $(Z_1 - sY_1)_{11}$ *in the canonical form of such pencils causes the corresponding differences from the rules above.*

7 A bundle stratification example

To illustrate the presented stratification rules we consider a system pencil $\mathcal{S}$ on the form (2) associated with a (generalized) state-space system consisting of $n = 5$ states, $m = 3$ inputs, and $p = 3$ outputs. The bundle closure hierarchy graph of such a system was computed by StratiGraph [22,24] and a subgraph of the complete stratification is shown in Figure 2 (the complete graph has 685 nodes and 2176 edges). Each node in the graph represents a bundle of

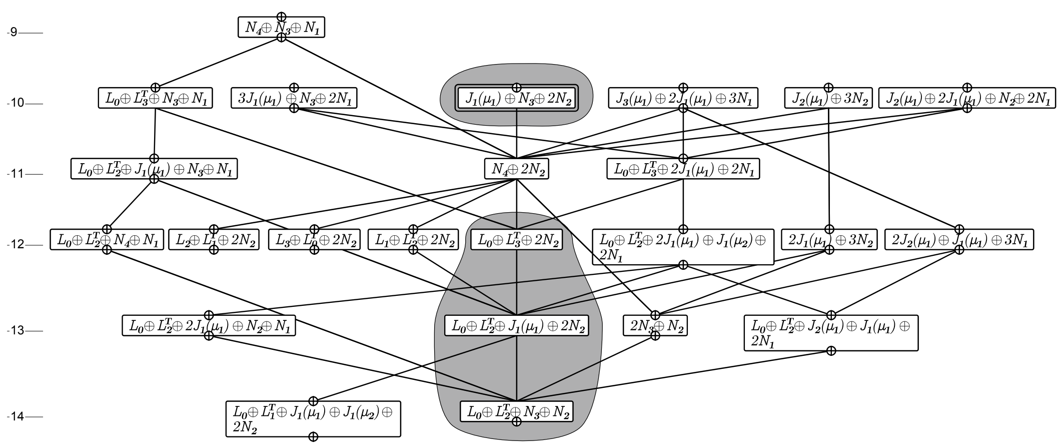

Fig. 2. A subgraph of the bundle closure hierarchy of a system pencil corresponding to a non-singular generalized state-space system with 5 states, 3 inputs, and 3 outputs. A circle with a plus on a node's border indicates that possibly not all closest neighbours are shown. By clicking on such a node-plus the graph is further expanded.

a system pencil under feedback-injection equivalence with the canonical form stated in the node, each edge is a cover relation between two bundles, and on the left the codimensions of the bundles are listed. Notably, in StratiGraph, the canonical form is presented using an abbreviated form which we also use in this section. In particular, the canonical blocks under feedback-injection (11) are defined as:

$$L_\epsilon := (G_\epsilon - sH_\epsilon)_{01}, \qquad L_\eta^T := (G_\eta^T - sH_\eta^T)_{10},$$
$$J_{h_i}(\mu_j) := (J_{h_i}(\mu_j) - sI_{h_i})_{00}, \text{ and } \quad N_q := (Z_q - sY_q)_{11}.$$

The hierarchical relation between the bundles should be interpreted as follows. It is always possible by an arbitrary small perturbation of any pencil in a bundle to obtain a pencil belonging to any bundle higher up in the graph, i.e. corresponding to a more generic system pencil, if they are connected by a path directed strictly upward. To go from a bundle to another below in the graph, i.e. to a more degenerate (less generic) case, in general requires a relative large perturbation of the system pencil entries.

What changes of the system characteristics can be investigated from the stratification in Figure 2? Below, we exemplify with some of the characteristics discussed in Section 2.2.

The changes of the finite invariant zeros μ_j of orders h_i, $i, j \in \{1, 2, \ldots\}$, are reflected by the $J_{h_i}(\mu_j)$ blocks, and the changes of the infinite zeros of orders $q_i - 1$, $i = 1, 2, \ldots$, by the N_{q_i} blocks, $q_i > 1$. The bundles with no N_1 blocks correspond to strictly proper transfer functions ($\mathcal{S}$ with $D = 0$), while the complete graph encloses both $D \neq 0$ and $D = 0$. Notably, maximum rank of D is three and therefore at most three N_1 block may exist. If $\mathcal{S}$ is a minimal realization then the finite zeros are transmission zeros, otherwise further analysis is needed to determine the types of the zeros. For example, the input- and output-decoupling zeros (uncontrollable and unobservable modes) can be analyzed by studying the controllability and observability system pencils, respectively [17].

To illustrate how the stratification can be used we consider a decoupling example with input failure taken from [14]. Our intention of studying this example is not to solve the problem of decoupling. Instead we show how the stratification can be used as a qualitative tool providing insight into how the different input failures change the system characteristics and how the corresponding system pencils relate in the closure hierarchy. Briefly, the purpose of decoupling is to separate a complex multi-input/multi-output (MIMO) system into several smaller subsystems that are easier to analyze and design controllers for. If a system can be *diagonally decoupled* it is possible to regard the system as a set of single-input/single-output (SISO) systems. Only systems with equal number of inputs and outputs ($m = p$) or more inputs than outputs ($m > p$) may be diagonally decoupled. If diagonal decoupling is not possible, it may still be possible to reduce the system to a set of independent (decoupled) subsystems, called *block decoupled*. For further information on decoupling and definitions we refer to [29,31] and references therein.

The system matrices ($n = 5, m = p = 3$) of the strictly proper system examined in [14] are

$$E = I_5, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad D = 0_{3\times 3}, \tag{25}$$

with the canonical form $J_1(-2) \oplus N_3 \oplus 2N_2$. The system is both controllable and observable so the finite eigenvalue is a transmission zero (it is of minimum phase). The system is both left and right invertible and can be diagonally decoupled. Figure 3 shows a subgraph of the stratification of the associated controllability pencil $\begin{bmatrix} A - sI & B \end{bmatrix}$, which belongs to the most generic bundle with the canonical form $2L_2 \oplus L_1$ (top node) when A and B are given by (25).

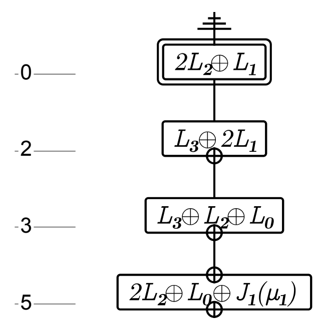

Fig. 3. A subgraph of the bundle closure hierarchy of a controllability pencil $[A - sI \ B]$ with 5 states and 3 inputs.

We now investigate how the canonical structure changes by imposing failure to one of the three inputs, for example, caused by loss of an actuator. The failure is introduced by setting the corresponding column in B to zero which will introduce a column minimal index of order zero (an L_0 block). By neglecting this redundant input, we get a non-square system with one more output than inputs ($m < p$) that cannot be diagonally decoupled. However, (as we will see) the perturbed systems are all left invertible so they may (but not for sure) be block decoupled [14,30]. Notably, all nodes in Figure 2 with one L_0 block correspond to a system with one redundant input. In the following we focus on the nodes in the grayed areas in Figure 2, where the corresponding bundle of system (25) is the top node. The remaining bundles outside of the grayed areas correspond to systems which are not of interest for our example. They can only be reached if other types of perturbations are allowed. Remember that in the stratification shown in Figure 2 changes in the D matrix are allowed.

Failure in the first input results in a system with the canonical form $L_0 \oplus L_2^T \oplus J_1(-2) \oplus 2N_2$ (codimension 13). Failure in the second input results in the canonical form $L_0 \oplus L_2^T \oplus N_3 \oplus N_2$ (codimension 14). Both these systems are controllable and can be block decoupled; they belong to the bundle $L_3 \oplus L_2 \oplus L_0$

in Figure 3. Failure in the third input results in a system that is still left invertible with the canonical form $L_0 \oplus L_3^T \oplus 2N_2$ (codimension 12 in Figure 2), but it is uncontrollable with the canonical form $2L_2 \oplus L_0 \oplus J_1(-1)$ and thus it is not possible to decouple the system [14].

An interesting observation is that failure in input one or two results in a less generic canonical form of the system pencil (2) associated with (25), than failure in input three that cannot be decoupled, see Figure 2. However, by looking at the corresponding bundles in the stratification of the controllability pencils in Figure 3, we see that the resulting system with failure in input three is less generic. A conclusion is that, along with the stratification of the system pencil we often need either the stratification of the controllability or the observability pencil (or both) to get the complete picture.

Acknowledgements

The work was supported by the Swedish Research Council (VR) under grant E0485301, and by eSSENCE (essenceofescience.se), a strategic collaborative e-Science programme funded by the Swedish Government via VR.

Appendix A: Upward coin moves

For a given system pencil orbit or bundle, Theorems 15 and 17, respectively, in Section 6 provide the rules to find the neighbours below in the stratification. Nevertheless, it may be more convenient or even necessary to know the neighbours above in a given pencil stratification. The following two theorems provide the rules in forms of upward coin moves to find the neighbours above for given system pencils in the stratifications.

Theorem 18 $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_1}$ is covered by $\mathcal{O}^{\text{f-i}}_{\mathcal{S}_2}$ *if and only if* $\mathcal{S}_2$ *can be obtained by applying one of the rules (1)–(6) to the structure integer partitions of* $\mathcal{S}_1$*:*

(1) Minimum leftward coin move in $\mathcal{R}$ *(or* $\mathcal{L}$*).*
(2) If $\mathcal{R}$ *(or* $\mathcal{L}$*) is non-empty and the rightmost column in* $\mathcal{J}_{\mu_i}$ *is one single coin, move that coin to a new rightmost column of* $\mathcal{R}$ *(or* $\mathcal{L}$*).*
(3) If $\mathcal{R}$ *(or* $\mathcal{L}$*) is non-empty and the rightmost column in* $\mathcal{Z}$ *is one single coin, move that coin to a new rightmost column of* $\mathcal{R}$ *(or* $\mathcal{L}$*). The rule cannot be applied if* $\mathcal{Z}$ *consists of only one coin, i.e.,* $\mathcal{Z}$ *cannot disappear after applying this rule.*
(4) Minimum rightward coin move in any $\mathcal{J}_{\mu_i}$*.*

(5) *Minimum rightward coin move in $\mathcal{Z}$, but no moves from the first (leftmost) column are allowed.*
(6) *If both $\mathcal{R}$ and $\mathcal{L}$ are non-empty: Let k denote the total number of coins in all of the longest (= lowest) rows from both $\mathcal{R}$ and $\mathcal{L}$ together. Remove these k coins, subtract one coin from the set, and distribute $k-1$ coins as follows. If $\mathcal{Z}$ is empty then give it one coin, otherwise add one coin to each nonzero column in $\mathcal{Z}$. Then distribute one coin to each nonzero column in all existing $\mathcal{J}_{\mu_i}$. The remaining coins are distributed among new rightmost columns, with one coin per column, to the existing $\mathcal{Z}$ or to any $\mathcal{J}_{\mu_i}$ which may be empty initially (i.e., new partitions for new finite eigenvalues can be created).*

Rules (1)–(2) are not allowed to do coin moves that affect r_0 or l_0 (first column in $\mathcal{R}$ and $\mathcal{L}$, respectively). Rule (6) cannot be applied if the total number of nonzero columns of $\mathcal{Z}$ and $\mathcal{J}_{\mu_i}$ are more than $k-1$.

PROOF. One can notice that these rules are the "reversed" rules from Theorem 15 or explicitly perform a proof analogous to the proof of Theorem 15.

Theorem 19 $\mathcal{B}^{\text{f-i}}_{\mathcal{S}_1}$ is covered by $\mathcal{B}^{\text{f-i}}_{\mathcal{S}_2}$ *if and only if $\mathcal{S}_2$ can be obtained by applying one of the rules (1)–(6) to the structure integer partitions of $\mathcal{S}_1$:*

(1) *Minimum leftward coin move in $\mathcal{R}$ (or $\mathcal{L}$).*
(2) *If $\mathcal{R}$ (or $\mathcal{L}$) is non-empty and $\mathcal{J}_{\mu_i}$ consists of one single coin, move that coin to a new rightmost column of $\mathcal{R}$ (or $\mathcal{L}$).*
(3) *Not applicable.*
(4) *Minimum rightward coin move in any $\mathcal{J}_{\mu_i}$.*
(5) *Minimum rightward coin move in $\mathcal{Z}$ but no moves from the first column are allowed.*
(6) *If both $\mathcal{R}$ and $\mathcal{L}$ are non-empty: Take one coin from each nonzero column of $\mathcal{R}$ and $\mathcal{L}$, remove one coin from the received amount, and distribute rest of the coins as follows. If $\mathcal{Z}$ is empty then give it one coin otherwise give one coin to each existing column in $\mathcal{Z}$. Then distribute one coin to each existing column in every $\mathcal{J}_{\mu_i}$. The remaining coins distribute among the new columns for the existing $\mathcal{J}_{\mu_i}, \mathcal{Z}$.*
(7) *Split $\mathcal{J}_{\mu_i}$ into two new partitions corresponding to two different finite eigenvalues.*
(8) *Split $\mathcal{Z}$ into two new partitions one corresponding to a new finite and the other to the infinite eigenvalue such that the number of blocks corresponding to infinite eigenvalue, i.e., z_0, remains the same.*

Rules (1)–(2) may not make coin moves that affect r_0 (or l_0).

PROOF. As for the orbit case, these rules are the "reversed" rules from Theorem 17.

References

[1] P. Antsaklis and A. Michel. *Linear Systems.* Birkhäuser, Boston, 2nd edition, 2006.

[2] V.I. Arnold. On matrices depending on parameters. *Russian Math. Surveys*, 26:29–43, 1971.

[3] V.I. Arnold. Lectures on bifurcations in versal families. *Russian Math. Surveys*, 27(5):54–123, 1972.

[4] V.I. Arnold. *Geometrical methods in the theory of ordinary differential equations.* Springer-Verlag, New York, 2nd edition, 1997.

[5] J.M. Berg and H.G. Kwatny. Unfolding the zero structure of a linear control system. *Linear Algebra Appl.*, 258:19–39, 1997.

[6] B.M. Chen, Z. Lin, and Y. Shamash. *Linear systems theory: A structural decomposition approach.* Birkhäuser, Boston, 2004.

[7] I. de Hoyos. Points of continuity of the Kronecker canonical form. *SIAM J. Matrix Anal. Appl.*, 11(2d):278–300, 1990.

[8] A. Dmytryshyn. Structure preserving stratification of skew-symmetric matrix polynomials. *Report UMINF 15.16, Dept. of Computing Science, Umeå University, Sweden*, 2015.

[9] A. Dmytryshyn, V. Futorny, B. Kågström, L. Klimenko, and V.V. Sergeichuk. Change of the congruence canonical form of 2-by-2 and 3-by-3 matrices under perturbations and bundles of matrices under congruence. *Linear Algebra and its Applications*, 469:305–334, 2015.

[10] A. Dmytryshyn, V. Futorny, and V.V. Sergeichuk. Miniversal deformations of matrices under *congruence and reducing transformations. *Linear Algebra Appl.*, 446:388–420, 2014.

[11] A. Dmytryshyn, S. Johansson, B. Kågström, and P. Van Dooren. Geometry of spaces for matrix polynomial fiedler linearizations. *Report UMINF 15.17, Dept. of Computing Science, Umeå University, Sweden*, 2015.

[12] A. Dmytryshyn and B. Kågström. Orbit closure hierarchies of skew-symmetric matrix pencils. *SIAM J. Matrix Anal. Appl.*, 35(4):1429–1443, 2014.

[13] A.R. Dmytryshyn, V. Futorny, and V.V. Sergeichuk. Miniversal deformations of matrices of bilinear forms. *Linear Algebra Appl.*, 436:2670–2700, 2012.

[14] P. Dworak. Squaring down plant model and I/O grouping strategies for a dynamic decoupling of left-invertible MIMO plants. *Bulletin of the Polish Academy of sciences technical sciences*, 62(3):471–479, 2014.

[15] A. Edelman, E. Elmroth, and B. Kågström. A geometric approach to perturbation theory of matrices and matrix pencils. Part I: Versal deformations. *SIAM J. Matrix Anal. Appl.*, 18(3):653–692, 1997.

[16] A. Edelman, E. Elmroth, and B. Kågström. A geometric approach to perturbation theory of matrices and matrix pencils. Part II: A stratification-enhanced staircase algorithm. *SIAM J. Matrix Anal. Appl.*, 20(3):667–669, 1999.

[17] E. Elmroth, S. Johansson, and B. Kågström. Stratification of controllability and observability pairs — Theory and use in applications. *SIAM J. Matrix Anal. Appl.*, 31(2):203–226, 2009.

[18] F. Gantmacher. *The theory of matrices, Vol. I and II (transl.)*. Chelsea, New York, 1959.

[19] M.I. García-Planas. Kronecker stratification of the space of quadruples of matrices. *SIAM J. Matrix Anal. Appl.*, 19(4):872–885, 1998.

[20] M.I García-Planas and M.D. Magret. Associating matrix pencils to generalized linear multivariable systems. *Linear Algebra Appl.*, 332:235–256, 2001.

[21] R.A. Horn and C.R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, 1985.

[22] P. Johansson. *Software Tools for Matrix Canonical Computations and Web-based Software Library Environments*. PhD thesis, Department of Computing Science, Umeå University, Sweden, November 2006.

[23] S. Johansson, B. Kågström, and P. Van Dooren. Stratification of full rank polynomial matrices. *Linear Algebra Appl.*, 439:1062–1090, 2013.

[24] B. Kågström, S. Johansson, and P. Johansson. StratiGraph Tool: Matrix Stratification in Control Applications. In L.T. Biegler, S.L. Campbell, and V. Mehrmann, editors, *Control and Optimization with Differential-Algebraic Constraints*, chapter 5. SIAM Publications, 2012.

[25] B.P. Molinari. Structural invariants of linear multivariable systems. *Internat. J. Control*, 28(4):493–510, 1978.

[26] P.J. Moylan. Stable inversion of linear systems. *IEEE Transactions on Automatic Control*, 22(1):74–78, 1977.

[27] J.S. Thorp. The singular pencil of a linear dynamical system. *Internat. J. Control*, 18(3):577–596, 1973.

[28] G.C. Verghese, B.C. Lévy, and T. Kailath. A generalized state-space for singular systems. *IEEE T. Automat. Contr.*, AC-26:811–830, 1981.

[29] Qing-Guo Wang. *Decoupling control*, volume 285. Springer, 2003.

[30] T. Williams and P.J. Antsaklis. A unifying approach to the decoupling of linear multivariable systems. *Int. J. Control*, 44(1):181–201, 1986.

[31] T. Williams and P.J. Antsaklis. Decoupling. In *The Control Handbook: Control System Advanced Methods*, chapter 27, pages 795–804. CRC Press, 2nd edition, 2010. Volume 3, Section III.