The solution of a pair of matrix equations $(X^{T}A+AX,\ X^{T}B+BX)=(0,0)$ with skew-symmetric A and B

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UMINF-12/05

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The solution of a pair of matrix equations $(X^TA + AX, X^TB + BX) = (0, 0)$ with skew-symmetric A and B *

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Abstract

The homogeneous system of matrix equations $(X^TA+AX, X^TB+BX) = (0, 0)$, where (A, B) is a pair of skew-symmetric matrices of the same size is considered: we establish the general solution and calculate the codimension of the orbit of (A, B) under congruence.

AMS classification: 15A24, 15A21

Keywords: Pair of skew-symmetric matrices; Matrix equations; Orbits; Codimension

1 Introduction

The work is inspired by a paper of De Terán and Dopico [3], where the general solution of the matrix equation $X^TA + AX = 0$ for a general square matrix A is derived. In the subsequent article [4], they established similar results for the matrix equation $XA + AX^* = 0$. Generalizing these results, De Terán et al. [5] found the general solutions of $AX + X^TB = 0$ and $AX + X^*B = 0$, respectively. These equations are homogeneous versions of the Sylvester equations for congruence and *congruence, respectively, which are

*A preprint appears as **Report UMINF 12.05**.

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important due to their relationship with palindromic eigenvalue problems (for more detail about motivations and applications of these problems see the introductions of [3, 4, 5] and the references therein).

Our objective is to present the general solution of the homogeneous system of matrix equations

$$\begin{aligned} X^T A + AX &= 0, \\ X^T B + BX &= 0, \end{aligned} \tag{1}$$

where (A, B) is a pair of skew-symmetric $n \times n$ matrices. The set of matrices X that satisfies the system (1) form a vector space whose dimension is also calculated. Indeed, this dimension minus n is equal to the codimension of the orbit of (A, B) (see Section 3.1).

Since the set $\{V^T(A, B) + (A, B)V) | V \in \mathbb{C}^{n \times n}\}$ is the tangent space to the congruence orbit of (A, B) at the point (A, B) the result is relevant to the theory of deformations of matrices and system pencils. This theory was created by V.I. Arnold (e.g., see [1]) and has been actively developing in the last years. In particular, deformations of pairs of skew-symmetric matrices are given in [9]. For more references about deformation theory, see [11, 17, 10, 7, 8] and references therein.

Note also that explicit expressions for codimensions of orbits were obtained recently for the following cases: matrix pencils [6], congruence orbits of matrices [3, 7], *congruence orbits of matrices [4, 8], congruence orbits of pairs of symmetric matrices [10], and generalized matrix products [22].

Both deformation theory and dimension/codimension calculations are useful in the theory of orbits and their stratifications (i.e., constructing the closure hierarchies; e.g., see [12, 13, 14, 20, 21, 19] for more details, algorithms, and software):

- the theory developed in [11] is used for the stratification of orbits of matrix pencils, controllability and observability pairs in [12, 14];
- the theory developed in [7] is used for the stratification of orbits of matrices of bilinear forms (for small dimensions) in [15].

Using the result of this paper Matlab functions for computing codimensions of congruence orbits of skew-symmetric matrix pencils were developed and added to the Matrix Canonical Structure Toolbox [18].

The analogous motivation and importance of symmetric matrices bring us to the problem (1) in which (A, B) is a pair of symmetric $n \times n$ matrices. This is a part of ongoing research.

The rest of the paper is organized as follows. The main results are presented in Section 2. Without loss of generality, we consider a congruently transformed system (1) where the skew-symmetric pair (A, B) is in canonical form. The general solution of the system (1) in explicit form is given in Theorem 2.1. The dimensions of solution spaces and codimensions of orbits are given in Corollary 2.1 and Corollary 2.2. In Section 3, we prove Theorem 2.1, Corollary 2.1, and Corollary 2.2. First a general result about codimension calculations is established in Section 3.1. In Sections 3.2–3.3, the nine different cases for the solution of the (transformed) system of matrix equations are handled. We end by illustrating the results on two pairs of matrix equations in canonical form in Section 4.

All matrices that we consider are over the field of complex numbers.

2 Main result

A pair (A, B) is said to be *congruent* to (A', B') if $(A', B') = S^T(A, B)S = (S^TAS, S^TBS)$ for some nonsingular S. Multiplying the equations (1) by S^T and S, we obtain

$$\begin{aligned} S^TX^TS^{-T} \cdot S^TAS + S^TAS \cdot S^{-1}XS &= 0, \\ S^TX^TS^{-T} \cdot S^TBS + S^TBS \cdot S^{-1}XS &= 0, \end{aligned}$$

and so the system (1) is equivalent to the system

$$\begin{aligned} Y^TA' + A'Y &= 0, \\ Y^TB' + B'Y &= 0, \end{aligned} \tag{2}$$

where $Y \equiv S^{-1}XS$. Therefore, it suffices to solve the system (1) in which (A, B) is a canonical pair of skew-symmetric matrices up to congruence.

Define the matrices

$$J_n(\lambda) := \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \quad (n\text{-by-}n \text{ Jordan block}),$$

$$F_n := \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}, \qquad G_n := \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix} \quad (n\text{-by-}(n+1)),$$

and define the direct sum of matrix pairs as follows:

$$(A, B) \oplus (C, D) = (A \oplus C, B \oplus D).$$

A canonical form of a pair of skew-symmetric matrices is given in the following lemma.

Lemma 2.1 (see [23]). *Every pair of skew-symmetric complex matrices is congruent to a direct sum, determined uniquely up to permutation of summands, of pairs of the form*

$$H_n(\lambda) := \left(\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} \right), \qquad \lambda \in \mathbb{C}, \tag{3}$$

$$K_n := \left(\begin{bmatrix} 0 & J_n(0) \\ -J_n(0)^T & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \right), \tag{4}$$

$$L_n := \left(\begin{bmatrix} 0 & F_n \\ -F_n^T & 0 \end{bmatrix}, \begin{bmatrix} 0 & G_n \\ -G_n^T & 0 \end{bmatrix} \right). \tag{5}$$

Thus, each pair of skew-symmetric matrices is congruent to a direct sum of the form

$$(A, B)_{\text{can}} = \bigoplus_{i=1}^{a} H_{h_i}(\lambda_i) \oplus \bigoplus_{j=1}^{b} K_{q_j} \oplus \bigoplus_{l=1}^{c} L_{r_l}, \tag{6}$$

consisting of direct summands of three types.

In the following, we define several parameter matrices, whose nonzero entries $p_1, p_2, p_3, \ldots$ are independent parameters; they will be used to express the set of solutions of (1).

- The $m \times n$ Hankel matrices

$$P_{mn} := \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & & \\ p_2 & p_3 & p_4 & & \cdot & \\ p_3 & p_4 & & \cdot & & \\ p_4 & & \cdot & & & \\ & \cdot & & & & \\ & & & & & p_{m+n-1} \end{bmatrix} \quad \text{(a dense Hankel matrix)},$$

$$P^{\nwarrow}_{mn} := \begin{bmatrix} p_1 & \dots & p_m & 0 & \\ \vdots & \iddots & \iddots & & \\ p_m & 0 & & & 0 \end{bmatrix} \text{ if } m \leqslant n \text{ and } \begin{bmatrix} p_1 & \dots & p_n \\ \vdots & \iddots & 0 \\ p_n & \iddots & \\ 0 & & \\ & & 0 \end{bmatrix} \text{ if } m > n,$$

$$P^{\Nwarrow}_{mn} := \begin{bmatrix} p_1 & \dots & p_m & 0 & \\ \vdots & \iddots & \iddots & & \\ p_m & 0 & & & 0 \end{bmatrix} \text{ if } m \leqslant n \text{ and } \begin{bmatrix} p_1 & \dots & p_n \\ \vdots & \iddots & \\ p_n & & \vdots \\ \vdots & & p_m \\ & \iddots & \\ p_m & & 0 \end{bmatrix} \text{ if } m > n.$$

- The matrices $P^{\nearrow}_{mn}$ and $P^{\Nearrow}_{mn}$ are obtained from $P^{\nwarrow}_{mn}$ and $P^{\Nwarrow}_{mn}$ by reflection with respect to the vertical axis. The matrices $P^{\searrow}_{mn}$ and $P^{\Searrow}_{mn}$ are obtained from $P^{\nearrow}_{mn}$ and $P^{\Nearrow}_{mn}$ by reflection with respect to the horizontal axis. The matrices $P^{\swarrow}_{mn}$ and $P^{\Swarrow}_{mn}$ are obtained from $P^{\nwarrow}_{mn}$ and $P^{\Nwarrow}_{mn}$ by reflection with respect to the horizontal axis. (Thus, each of these matrices is constructed like $P^{\nwarrow}_{mn}$ and $P^{\Nwarrow}_{mn}$ but its parameter diagonals are disposed in the corner pointed to by the arrow.)

- The $m \times n$ banded Toeplitz matrices

$$P^{\leftrightarrow}_{mn} := \begin{bmatrix} p_1 & \dots & p_{n-m+1} & & & 0 \\ & \ddots & & & \ddots & \\ 0 & & & p_1 & \dots & p_{n-m+1} \end{bmatrix} \text{ if } m \leqslant n \text{ and } 0 \text{ if } m > n,$$

$$P^{\updownarrow}_{mn} := 0 \text{ if } m < n \text{ and } \begin{bmatrix} p_1 & & 0 \\ \vdots & \ddots & \\ & & p_1 \\ p_{m-n+1} & & \\ & \ddots & \vdots \\ 0 & & p_{m-n+1} \end{bmatrix} \text{ if } m \geqslant n.$$

- The $(m+1) \times n$ matrices defined recurrently

$$P^{\mathrm{lc}}_{m+1,n}(\lambda) = \begin{cases} p_{in} = \frac{\alpha_1}{\lambda^{m-i+1}}, & 1 \leqslant i \leqslant m+1 \text{ (last column (lc))}, \\ p_{m+1,j} = \alpha_{n-j+1}, & 1 \leqslant j \leqslant n \text{ (last row)}, \\ p_{ij} = \frac{1}{\lambda}(p_{i+1j} - p_{ij+1}), & 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant n-1, \end{cases}$$

and

$$P^{\mathrm{fc}}_{m+1,n}(\lambda) = \begin{cases} p_{i1} = \frac{\alpha_1}{\lambda^{m+1-i}}, & 1 \leqslant i \leqslant m+1 \text{ (first column (fc))}, \\ p_{m+1,j} = \alpha_j, & 1 \leqslant j \leqslant n \text{ (last row)}, \\ p_{ij} = \frac{1}{\lambda}(p_{i+1j} - p_{ij-1}), & 1 \leqslant i \leqslant m \text{ and } 2 \leqslant j \leqslant n, \end{cases}$$

where $\lambda \in \mathbb{C}\backslash\{0\}$.

Moreover, we denote by

$$Q_{mn}, Q^{\nwarrow}_{mn}, Q^{\nwarrow\!\!\nwarrow}_{mn}, \dots, \qquad R_{mn}, R^{\nwarrow}_{mn}, R^{\nwarrow\!\!\nwarrow}_{mn}, \dots, \qquad S_{mn}, S^{\nwarrow}_{mn}, S^{\nwarrow\!\!\nwarrow}_{mn}, \dots$$

the parameter matrices that are obtained from $P_{mn}, P^{\nwarrow}_{mn}, P^{\nwarrow\!\!\nwarrow}_{mn}, \dots$ by replacing all parameters p_i with q_i, r_i, and s_i, respectively.

Note that when two (or more) matrices are denoted by the same letter, they have exactly the same set of independent parametric entries (regardless to the sizes of matrices) and the parametric entries are placed according to the definitions above.

We say that a parameter matrix $P \equiv P(\varepsilon_1, \dots, \varepsilon_s)$ with independent parameters $\varepsilon_1, \dots, \varepsilon_s$ is *isomorphic* to a parameter matrix $Q \equiv Q(\delta_1, \dots, \delta_s)$ with independent parameters $\delta_1, \dots, \delta_s$ and write

$$P(\varepsilon_1, \dots, \varepsilon_s) \simeq Q(\delta_1, \dots, \delta_s)$$

if they coincide up to relettering of parameters; that is, if there exists a permutation σ of $\{1, \dots, s\}$ such that $P(\delta_{\sigma(1)}, \dots, \delta_{\sigma(s)}) = Q(\delta_1, \dots, \delta_s)$.

Let (A, B) be a canonical matrix pair and let

$$(A, B) = (A_1, B_1) \oplus \dots \oplus (A_t, B_t), \qquad t := a + b + c,$$

be its decomposition (6). Let $\mathcal{P}$ be a parameter matrix that has the same size as A and B and the same partition into blocks:

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_{11} & \dots & \mathcal{P}_{1t} \\ \vdots & \ddots & \vdots \\ \mathcal{P}_{t1} & \dots & \mathcal{P}_{tt} \end{bmatrix}, \qquad \text{size } (\mathcal{P}_{ii}) = \text{size } (A_i) = \text{size } (B_i). \tag{7}$$

Write

$$\mathcal{P}((A_i, B_i)) := \mathcal{P}_{ii}, \qquad \mathcal{P}((A_i, B_i), (A_j, B_j)) := (\mathcal{P}_{ji}, \mathcal{P}_{ij}) \text{ if } i < j. \tag{8}$$

The canonical pair (6) and the following conditions determine $\mathcal{P}$ uniquely up to isomorphism:

(i) If $\mathcal{P}_{ij}$ and $\mathcal{P}_{i'j'}$ have overlapping sets of parameters, then $i' = j$ and $j' = i$.

(ii) The diagonal blocks of $\mathcal{P}$ are defined up to isomorphism by

$$\mathcal{P}(H_n(\lambda)) \simeq \begin{bmatrix} -P_{nn}^{\swarrow} & R_{nn}^{\searrow} \\ Q_{nn}^{\nwarrow} & P_{nn}^{\nearrow} \end{bmatrix}, \tag{9}$$

$$\mathcal{P}(K_n) \simeq \begin{bmatrix} -P_{nn}^{\swarrow} & R_{nn}^{\searrow} \\ Q_{nn}^{\nwarrow} & P_{nn}^{\nearrow} \end{bmatrix}, \tag{10}$$

$$\mathcal{P}(L_n) \simeq \begin{bmatrix} -\alpha I_n & 0_{n,n+1} \\ P_{n+1,n} & \alpha I_{n+1} \end{bmatrix}. \tag{11}$$

(iii) The off-diagonal blocks of $\mathcal{P}$ whose horizontal and vertical strips contain summands of $(A,B)_{\text{can}}$ of the same type are defined up to isomorphism by

$$\mathcal{P}(H_n(\lambda),\, H_m(\mu)) \simeq \begin{cases} (0,\, 0) & \text{if } \lambda \neq \mu, \\ \left(\begin{bmatrix} P_{mn}^{\swarrow} & R_{mn}^{\searrow} \\ Q_{mn}^{\nwarrow} & S_{mn}^{\nearrow} \end{bmatrix}, \begin{bmatrix} -S_{nm}^{\swarrow} & R_{nm}^{\searrow} \\ Q_{nm}^{\nwarrow} & -P_{nm}^{\nearrow} \end{bmatrix} \right) & \text{if } \lambda = \mu, \end{cases} \tag{12}$$

$$\mathcal{P}(K_n, K_m) \simeq \left(\begin{bmatrix} P_{mn}^{\swarrow} & R_{mn}^{\searrow} \\ Q_{mn}^{\nwarrow} & S_{mn}^{\nearrow} \end{bmatrix}, \begin{bmatrix} -S_{nm}^{\swarrow} & R_{nm}^{\searrow} \\ Q_{nm}^{\nwarrow} & -P_{nm}^{\nearrow} \end{bmatrix} \right), \tag{13}$$

$$\mathcal{P}(L_n, L_m) \simeq \left(\begin{bmatrix} P_{nm}^{\leftrightarrow T} & 0_{m,n+1} \\ Q_{m+1,n} & R_{n+1,m+1}^{\updownarrow T} \end{bmatrix}, \begin{bmatrix} R_{nm}^{\updownarrow} & 0_{n,m+1} \\ Q_{n+1,m} & P_{n+1,m+1}^{\leftrightarrow} \end{bmatrix} \right). \tag{14}$$

(iv) The off-diagonal blocks of $\mathcal{P}$ whose horizontal and vertical strips contain summands of $(A,B)_{\text{can}}$ of different types are defined up to isomorphism by

$$\mathcal{P}(H_n(\lambda), K_m) \simeq (0,\, 0), \tag{15}$$

$$\mathcal{P}(H_n(\lambda), L_m)$$
$$\simeq \begin{cases} \left(\begin{bmatrix} 0_{mn} & 0_{mn} \\ P^{\Nwarrow T}_{n,m+1} & Q^{\Swarrow T}_{n,m+1} \end{bmatrix}, \begin{bmatrix} -Q^{\Swarrow}_{n,m+1} F^T_m & 0_{n,m+1} \\ P^{\Nwarrow}_{n,m+1} F^T_m & 0_{n,m+1} \end{bmatrix} \right) & \text{if } \lambda = 0, \\ \left(\begin{bmatrix} 0_{mn} & 0_{mn} \\ P^{\mathrm{lc}}_{m+1,n}(\lambda) & Q^{\mathrm{fc}}_{m+1,n}(\lambda) \end{bmatrix}, \begin{bmatrix} -(F_m Q^{\mathrm{fc}}_{m+1,n}(\lambda))^T & 0_{n,m+1} \\ (F_m P^{\mathrm{lc}}_{m+1,n}(\lambda))^T & 0_{n,m+1} \end{bmatrix} \right) & \text{if } \lambda \neq 0, \end{cases} \tag{16}$$

$$\mathcal{P}(K_n, L_m) \simeq \left(\begin{bmatrix} 0_{mn} & 0_{mn} \\ P^{\Nearrow T}_{n,m+1} & Q^{\Searrow T}_{n,m+1} \end{bmatrix}, \begin{bmatrix} -Q^{\Searrow}_{n,m+1} G^T_m & 0_{n,m+1} \\ P^{\Nearrow}_{n,m+1} G^T_m & 0_{n,m+1} \end{bmatrix} \right). \tag{17}$$

Theorem 2.1. *Let the system* (1) *be given by the canonical pair* (6) *of skew-symmetric matrices for congruence. Let* $\mathcal{P}(\pi_1, \dots, \pi_s)$ *be a parameter matrix satisfying conditions* (i)–(iv)*. Then*

$$\{\mathcal{P}(a_1, \dots, a_s) \,|\, (a_1, \dots, a_s) \in \mathbb{C}^s\} \tag{18}$$

is the set of all solutions of the system (1)*.*

Corollary 2.1. *If the system* (1) *is given by the canonical pair* (6)*, then the dimension of its solution space* (18) *is equal to the sum*

$$d_{(A,B)} = d_H + d_K + d_L + d_{HH} + d_{KK} + d_{LL} + d_{HK} + d_{HL} + d_{KL} \tag{19}$$

whose summands correspond to

- *the direct summands of* (6)*:*

$$d_H \coloneqq 3 \sum_{i=1}^{a} p_i, \qquad d_K \coloneqq 3 \sum_{i=1}^{b} q_i, \qquad d_L \coloneqq c + 2 \sum_{i=1}^{c} r_i;$$

- *the pairs of direct summands of* (6) *of the same type:*

$$d_{HH} \coloneqq 4 \sum_{\substack{i \leqslant j \\ \lambda_i = \lambda_j}} \min(p_i, p_j), \qquad d_{KK} \coloneqq 4 \sum_{i \leqslant j} \min(q_i, q_j),$$

$$d_{LL} \coloneqq \sum_{j \leqslant i} \left(2 \max(r_i, r_j) + \varepsilon_{ij}\right), \qquad \textit{in which } \varepsilon_{ij} \coloneqq \begin{cases} 2 & \textit{if } r_i = r_j, \\ 1 & \textit{if } r_i \neq r_j; \end{cases}$$

- *the pairs of direct summands of* (6) *of different types:*

$$d_{HK} := 0, \qquad d_{HL} := 2\sum_{i,j} p_i, \qquad d_{KL} := 2\sum_{i,j} q_i.$$

The set of matrix pairs that are congruent to a pair (A,B) of skew-symmetric $n \times n$ matrices is a manifold in the complex $n^2 - n$ dimensional space of all pairs of skew-symmetric $n \times n$ matrices. This manifold is the orbit of (A,B) under the action of congruence. The vector space

$$T(A,B) := \{V^T(A,B) + (A,B)V \mid V \in \mathbb{C}^{n\times n}\} \tag{20}$$

is the tangent space to the congruence class of (A,B) at the point (A,B) since

$$\begin{aligned}&(I+\varepsilon V)^T(A,B)(I+\varepsilon V)\\ &\qquad = (A,B) + \varepsilon(V^T(A,B) + (A,B)V) + \varepsilon^2 V^T(A,B)V\end{aligned}$$

for all n-by-n matrices V and each $\varepsilon \in \mathbb{C}$. The *dimension of the orbit of* (A,B) is the dimension of its tangent space at the point (A,B); it is well defined because the dimensions of tangent spaces at all points of the orbit are equal (e.g., see [2]). The *codimension of the orbit of* (A,B) is the dimension of the normal space of its orbit at the point (A,B), which is equal to the dimension $n^2 - n$ of the space of all pairs of skew-symmetric $n \times n$ matrices minus the dimension of the orbit of (A,B).

Corollary 2.2. *The codimension of the congruence orbit of the canonical pair* (6) *of $n \times n$ skew-symmetric matrices is equal to*

$$d_{(A,B)} - n, \tag{21}$$

in which $d_{(A,B)}$ is the dimension (19) *of the solution space of system* (1).

Note that the codimensions of the orbits of canonical matrices A under congruence and *congruence are given in [7, 3] and [4]; unlike (21) they are exactly equal to the dimensions of the solution spaces of the equations $XA + AX^T = 0$ and $XA + AX^* = 0$.

3 Solution of the system of matrix equations

In this section we prove Theorem 2.1 and Corollary 2.1. Each direct canonical summand in (6) is of the form $H_n(\lambda)$, K_n, or L_n (see (3)–(5)), and so we need to determine:

- 3 types of the diagonals blocks of $\mathcal{P}$, each of them corresponds to one type of canonical summands (sections 3.2–3.3);
- 3 types of the off-diagonal blocks of $\mathcal{P}$, each of them corresponds to the different pairs of canonical summands of the same type (sections 3.4–3.5);
- 3 types of the off-diagonal blocks of $\mathcal{P}$, each of them corresponds to the different pairs of canonical summands of the different types (sections 3.6–3.8).

Corollary 2.2 is essentially restated and proved as Lemma 3.1 in Section 3.1.

3.1 On codimension computations

Let us state a general result about the codimension computations.

Theorem 3.1. *Let X and Z be finitely generated vector spaces and $X = Y \oplus N$. Then for any surjective linear map $f : Z \to Y$ we have that*

$$\dim N = \dim X - \dim Z + \dim \operatorname{Ker} f.$$

Proof. The proof follows immediately after we note that $\dim Z = \dim Y + \dim \operatorname{Ker} f$. □

Lemma 3.1. *The codimension of the orbit of $(A, B) \in \mathbb{C}_c^{n\times n} \times \mathbb{C}_c^{n\times n}$, where $\mathbb{C}_c^{n\times n}$ is the space of skew-symmetric $n \times n$ matrices, can be calculated as follows:*

$$\operatorname{codim}(\operatorname{orbit}(A, B)) = \dim V(A, B) - n$$

in which $V(A, B) := \{X \in \mathbb{C}^{n\times n} | X^T(A, B) + (A, B)X = 0\}$.

Proof. The result could be obtained from Theorem 3.1 but we give an independent proof. Define the function $f : \mathbb{C}^{n\times n} \to T(A,B)$ where $T(A,B)$ is a tangent space at the point (A,B) (see (20)) such that

$$X \mapsto X^T(A,B) + (A,B)X.$$

The mapping is obviously a surjective homomorphism thus $\dim \mathbb{C}^{n\times n} = \dim T(A,B) + \dim V(A,B)$. Also at every point $(A,B) \in \mathbb{C}_c^{n\times n} \times \mathbb{C}_c^{n\times n}$ we have the decomposition

$$\mathbb{C}_c^{n\times n} \times \mathbb{C}_c^{n\times n} = T(A,B) \oplus N(A,B),$$

where $N(A,B)$ is a normal space at the point (A,B). Therefore

$$\begin{aligned} \operatorname{codim}(\operatorname{orbit}(A,B)) &= \dim N(A,B) = \dim(\mathbb{C}_c^{n\times n} \times \mathbb{C}_c^{n\times n}) - \dim T(A,B) \\ &= \dim(\mathbb{C}_c^{n\times n} \times \mathbb{C}_c^{n\times n}) - \dim \mathbb{C}^{n\times n} + \dim V(A,B) = n^2 - n - n^2 + \dim V(A,B) \\ &= \dim V(A,B) - n. \end{aligned}$$

□

3.2 Solution for H and K blocks

In this section we solve the system (1) for $(A,B) = H_n(\lambda)$ and $(A,B) = K_n$. We start by considering $(A,B) = H_n(\lambda)$ and partition X conformally with the 2×2 block structure of $H_n(\lambda)$ and obtain the following system of matrix equations:

$$\begin{aligned} \begin{bmatrix} X_{11}^T & X_{21}^T \\ X_{12}^T & X_{22}^T \end{bmatrix} \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} X_{11}^T & X_{21}^T \\ X_{12}^T & X_{22}^T \end{bmatrix} \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \begin{bmatrix} X_{21} - X_{21}^T & X_{11}^T + X_{22} \\ -X_{11} - X_{22}^T & X_{12}^T - X_{12} \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} -X_{21}^T J_n(\lambda)^T + J_n(\lambda) X_{21} & X_{11}^T J_n(\lambda) + J_n(\lambda) X_{22} \\ -X_{22}^T J_n(\lambda)^T - J_n(\lambda)^T X_{11} & X_{12}^T J_n(\lambda) - J_n(\lambda)^T X_{12} \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{22}$$

This system decomposes into three different systems; each of them corresponds to one pair of blocks. Indeed, we have four pairs of blocks but two of them are equal up to the sign and transposition.

First consider the system corresponding to the $(1,2)$-blocks:

$$X_{11}^T + X_{22} = 0,$$
$$X_{11}^T J_n(\lambda) + J_n(\lambda) X_{22} = 0.$$

Note that

$$\begin{aligned} -X_{22}J_n(\lambda) + J_n(\lambda)X_{22} &= -\lambda X_{22} - X_{22}J_n(0) + J_n(0)X_{22} + \lambda X_{22} \\ &= -X_{22}J_n(0) + J_n(0)X_{22}. \end{aligned} \tag{23}$$

Analogously, we can transform the systems corresponding to the $(1,1)$- and $(2,2)$-blocks of (22), and therefore we can put $\lambda = 0$ in these systems.

Thus we have the equation

$$-X_{22}J_n(0) + J_n(0)X_{22} = 0, \tag{24}$$

and by [16, Ch. VIII] the solution $X_{22} = P_{nn}^{\nearrow}$ and therefore $X_{11} = -P_{nn}^{\nearrow T} = -P_{nn}^{\swarrow}$. Here and hereafter when we write that the unknown matrix is equal to a parametric matrix, we mean that any matrix obtained by replacing the parameters with any complex numbers is a solution and there is no solution of the system that can not be obtained in this way.

Now consider the system corresponding to the $(2,2)$-blocks:

$$X_{21} - X_{21}^T = 0,$$
$$-X_{21}^T J_n(0)^T + J_n(0)X_{21} = 0.$$

So we are looking for symmetric solutions of $-X_{21}J_n(0)^T + J_n(0)X_{21} = 0$. To solve this equation we use the solution of the first system. Multiplying (24) by the n-by-n flip matrix

$$Z := \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} \tag{25}$$

from the right hand side and taking into account that $Z^2 = I$ and $ZJ_n(0)Z = J_n(0)^T$ we get $-(X_{22}Z)J_n(0)^T + J_n(0)(X_{22}Z) = 0$. Taking into account the independence of the systems (hereafter we will usually skip this phrase in explanations) we have $X_{21} = X_{22}Z = Q_{nn}^{\nearrow}Z = Q_{nn}^{\nwarrow}$ which is already symmetric.

Finally, the system corresponding to the $(1,1)$-blocks is:

$$\begin{aligned} X_{12}^T - X_{12} &= 0, \\ X_{12}^T J_n(0) - J_n(0)^T X_{12} &= 0. \end{aligned}$$

Again we are looking for symmetric solutions of $X_{12}J_n(0) - J_n(0)^T X_{12} = 0$. Now we multiply (24) by $(-Z)$ but from the left hand side and get $(ZX_{22})J_n(0) - J_n(0)^T(ZX_{22}) = 0$. Thus we have $X_{12} = ZX_{22} = ZR_{nn}^{\nearrow} = R_{nn}^{\searrow}$ which is already symmetric. Altogether we obtain

$$X = \begin{bmatrix} -P_{nn}^{\swarrow} & R_{nn}^{\searrow} \\ Q_{nn}^{\nwarrow} & P_{nn}^{\nearrow} \end{bmatrix}.$$

Therefore the general solution of the system (22) is $X = \mathcal{P}(H_n(\lambda))$ (9). Since the solution does not depend on λ (see equation (23)), the system with $(A,B) = K_n$ has the same solution $X = \mathcal{P}(K_n)$ (10). We have $3n$ independent parameters in every of them therefore $d_K = d_H := \dim V(H_n(\lambda)) = \dim V(K_n) = 3n$ and by Corollary 2.2 $\operatorname{codim}(\operatorname{orbit} H_n(\lambda)) = \operatorname{codim}(\operatorname{orbit} K_n) = 3n - 2n = n$.

3.3 Solution for L blocks

In this section we solve the system (1) for $(A,B) = L_n$.

We partition X conformally with the 2×2 block structure of L_n and obtain the following system

$$\begin{bmatrix} X_{11}^T & X_{21}^T \\ X_{12}^T & X_{22}^T \end{bmatrix} \begin{bmatrix} 0 & F_n \\ -F_n^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & F_n \\ -F_n^T & 0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\begin{bmatrix} X_{11}^T & X_{21}^T \\ X_{12}^T & X_{22}^T \end{bmatrix} \begin{bmatrix} 0 & G_n \\ -G_n^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & G_n \\ -G_n^T & 0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

corresponding to the following pairs of blocks

$$\begin{aligned} \begin{bmatrix} -X_{21}^T F_n^T + F_n X_{21} & X_{11}^T F_n + F_n X_{22} \\ -X_{22}^T F_n^T - F_n^T X_{11} & X_{12}^T F_n - F_n^T X_{12} \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} -X_{21}^T G_n^T + G_n X_{21} & X_{11}^T G_n + G_n X_{22} \\ -X_{22}^T G_n^T - G_n^T X_{11} & X_{12}^T G_n - G_n^T X_{12} \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{26}$$

Since the pairs of blocks at positions $(1,2)$ and $(2,1)$ are equal up to the sign and transposition, (26) also decomposes into three different systems.

First consider the system corresponding to the $(1,1)$-blocks:

$$\begin{aligned} -X_{21}^T F_n^T + F_n X_{21} &= 0, \\ -X_{21}^T G_n^T + G_n X_{21} &= 0. \end{aligned} \tag{27}$$

To satisfy the first equation of (27) X_{21} has to have the following form

$$X_{21}^T = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} & x_{1n+1} \\ x_{12} & x_{22} & x_{23} & \dots & x_{2n} & x_{2n+1} \\ x_{13} & x_{23} & x_{33} & \dots & x_{3n} & x_{3n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{1n} & x_{2n} & x_{3n} & \dots & x_{nn} & x_{nn+1} \end{bmatrix}.$$

Substituting X_{21} into the second equation of (27) we have

$$\begin{aligned} &-\begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} & x_{1n+1} \\ x_{12} & x_{22} & x_{23} & \dots & x_{2n} & x_{2n+1} \\ x_{13} & x_{23} & x_{33} & \dots & x_{3n} & x_{3n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{1n} & x_{2n} & x_{3n} & \dots & x_{nn} & x_{nn+1} \end{bmatrix} \begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ & \ddots & 0 \\ 0 & & 1 \end{bmatrix} \\ &\qquad + \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{12} & x_{22} & x_{23} & \dots & x_{2n} \\ x_{13} & x_{23} & x_{33} & \dots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & x_{3n} & \dots & x_{nn} \\ x_{1n+1} & x_{2n+1} & x_{3n+1} & \dots & x_{nn+1} \end{bmatrix} = 0. \end{aligned}$$

Multiplying and identifying at entry level, we obtain

$$\begin{aligned} -x_{ij+1} + x_{i+1j} &= 0 \text{ if } i < j, \\ x_{ij+1} - x_{i+1j} &= 0 \text{ if } i > j, \\ 0 &= 0 \text{ if } i = j, \end{aligned}$$

where $i, j = 1, \dots, n$. Now it follows that $X_{21}^T = P_{nn+1}$ is the general solution of (27), which has $2n$ independent parameters.

Next consider the system corresponding to the $(1,2)$-blocks:

$$\begin{aligned} X_{11}^T F_n + F_n X_{22} &= 0, \\ X_{11}^T G_n + G_n X_{22} &= 0. \end{aligned} \tag{28}$$

To satisfy the first equation of (28), X_{22} must have the following form

$$X_{22} = \begin{bmatrix} & & 0 \\ & -X_{11}^T & \vdots \\ & & 0 \\ y_1 & \dots & y_{n+1} \end{bmatrix}, \tag{29}$$

where $X_{11}^T = [x_{ij}]$ is an $n \times n$ matrix. After substituting this value of X_{22} in the second equation of (28)

$$\begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nn} \end{bmatrix} \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix} \begin{bmatrix} -x_{11} & \dots & -x_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ -x_{n1} & \dots & -x_{nn} & 0 \\ y_1 & \dots & y_n & y_{n+1} \end{bmatrix} = 0,$$

we obtain

$$-\begin{bmatrix} x_{21} & -x_{11} + x_{22} & \dots & -x_{1n-1} + x_{2n} & -x_{1n} \\ x_{31} & -x_{21} + x_{32} & \dots & -x_{2n-1} + x_{3n} & -x_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ x_{n1} & -x_{n-11} + x_{n2} & \dots & -x_{nn-1} + x_{nn} & -x_{n-1n} \\ -y_1 & -x_{n1} - y_2 & \dots & -x_{nn-1} - y_n & -x_{nn} - y_{n+1} \end{bmatrix} = 0.$$

This means that $X_{22} = \alpha I_{n+1}$ and from (29) $X_{11} = -\alpha I_n$. The pair (X_{11}, X_{22}) is the general solution of the system (28) and it has only one parameter.

Now consider the system corresponding to the $(2,2)$-blocks:

$$\begin{aligned} X_{12}^T F_n - F_n^T X_{12} &= 0, \\ X_{12}^T G_n - G_n^T X_{12} &= 0. \end{aligned} \tag{30}$$

To satisfy the first equation of (30) X_{12} must have the following form

$$X_{12} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} & 0 \\ x_{12} & x_{22} & x_{23} & \dots & x_{2n} & 0 \\ x_{13} & x_{23} & x_{33} & \dots & x_{3n} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{1n} & x_{2n} & x_{3n} & \dots & x_{nn} & 0 \end{bmatrix}.$$

Substituting X_{12} into the second equation of (30) we obtain

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{12} & x_{22} & x_{23} & \dots & x_{2n} \\ x_{13} & x_{23} & x_{33} & \dots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & x_{3n} & \dots & x_{nn} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix}$$

$$- \begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ & \ddots & 0 \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} & 0 \\ x_{12} & x_{22} & x_{23} & \dots & x_{2n} & 0 \\ x_{13} & x_{23} & x_{33} & \dots & x_{3n} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{1n} & x_{2n} & x_{3n} & \dots & x_{nn} & 0 \end{bmatrix} = 0,$$

or equivalently

$$\begin{bmatrix} 0 & x_{11} & x_{12} & x_{13} & \dots & x_{1n-1} & x_{1n} \\ -x_{11} & 0 & x_{22}-x_{13} & x_{23}-x_{14} & \dots & x_{2n-1}-x_{1n} & x_{2n} \\ -x_{12} & x_{13}-x_{22} & 0 & x_{33}-x_{24} & \dots & x_{3n-1}-x_{2n} & x_{3n} \\ -x_{13} & x_{14}-x_{23} & x_{24}-x_{33} & 0 & \dots & x_{4n-1}-x_{3n} & x_{4n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -x_{1n-1} & x_{1n}-x_{2n-1} & x_{2n}-x_{3n-1} & x_{3n}-x_{4n-1} & \dots & 0 & x_{nn} \\ -x_{1n} & -x_{2n} & -x_{3n} & -x_{4n} & \dots & -x_{nn} & 0 \end{bmatrix} = 0.$$

It is clear that $X_{12} = 0$ (skew-diagonal-wise). Altogether we have

$$X = \begin{bmatrix} -\alpha I_n & 0_{n,n+1} \\ P_{n+1,n} & \alpha I_{n+1} \end{bmatrix}.$$

Thus the general solution of the system (26) is $X = \mathcal{P}(L_n)$ (11). We have $2n+1$ independent parameters in X thus $d_L := \dim V(L_n) = 2n+1$ and by Corollary 2.2 $\operatorname{codim}(\operatorname{orbit}(L_n)) = 2n+1-(2n+1) = 0$.

3.4 Interaction between $H_n(\lambda)$ and $H_m(\mu)$ blocks and between K_n and K_m blocks

Let us explain how (1) is changed when $(A,B) = (A_1,B_1) \oplus (A_2,B_2)$. It is enough to consider only the first matrix equation (the second is treated analogously):

$$\begin{bmatrix} X_1^T & X_3^T \\ X_2^T & X_4^T \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = 0,$$

or equivalently

$$\begin{bmatrix} X_1^T A_1 + A_1 X_1 & X_3^T A_2 + A_1 X_2 \\ X_2^T A_1 + A_2 X_3 & X_4^T A_2 + A_2 X_4 \end{bmatrix} = 0.$$

The off-diagonal blocks $X_3^T A_2 + A_1 X_2$ and $X_2^T A_1 + A_2 X_3$ are the same up to the skew-symmetry and it is enough to investigate just one of them.

In this section we calculate off-diagonal blocks of the solution of the system (1) when $A_1 = H_n(\lambda)$ and $A_2 = H_m(\mu)$. We remark that the system with $A_1 = K_n$ and $A_2 = K_m$ have the same solution and is therefore omitted in the discussion that follows.

We consider the following system of equations

$$\begin{aligned} R \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ R \begin{bmatrix} 0 & J_m(\mu) \\ -J_m(\mu)^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{31}$$

where R and S are $2n$-by-$2m$ matrices. In the notation at the beginning of this section $R = X_3^T$ and $S = X_2$; we change the notation to avoid over-indexing. The solution space of the system above is called the interaction between $H_n(\lambda)$ and $H_m(\mu)$ and denoted by $\operatorname{inter}(H_n(\lambda), H_m(\mu))$. The notion of interaction will be used (without analogous explanation) for all off-diagonal blocks.

From the first equation in (31) we have

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} -R_{22} & R_{21} \\ R_{12} & -R_{11} \end{bmatrix}.$$

By substituting this value of S in the second equation,

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} 0 & J_m(\mu) \\ -J_m(\mu)^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} \begin{bmatrix} -R_{22} & R_{21} \\ R_{12} & -R_{11} \end{bmatrix} = 0,$$

we obtain

$$\begin{bmatrix} -R_{12} J_m(\mu)^T + J_n(\lambda) R_{12} & R_{11} J_m(\mu) - J_n(\lambda) R_{11} \\ -R_{22} J_m(\mu)^T + J_n(\lambda)^T R_{22} & R_{21} J_m(\mu) - J_n(\lambda)^T R_{21} \end{bmatrix} = 0. \tag{32}$$

The matrix equation decomposes into four independent matrix equations. By [16, Ch.VIII] the equation $R_{11}J_m(\mu) = J_n(\lambda)R_{11}$ in the $(1,2)$-block has the solution $R_{11} = 0$ if $\lambda \neq \mu$ and $R_{11} = P^{\nearrow}_{nm}$ if $\lambda = \mu$.

All equations are independent and therefore we have to use different parameters to express the general solutions of the four blocks in (32).

Analogously to Section 3.2, we multiply the equations in (32) by the flip matrix Z (25) from right, left, or both sides. Using that $Z^2 = I$ and $ZJ(\lambda)Z = J(\lambda)^T$ we obtain the following general solutions for the remaining three equations:

$$(1,1)\text{-block:} \qquad -R_{12}J_m(\mu)^T + J_n(\lambda)R_{12} = R_{11}ZZJ_m(\mu)Z - J_n(\lambda)R_{11}Z$$

$$\text{therefore } R_{12} = -R_{11}Z = \begin{cases} 0 & \text{if } \lambda \neq \mu, \\ Q^{\nwarrow}_{nm} & \text{if } \lambda = \mu. \end{cases}$$

$$(2,2)\text{-block:} \qquad R_{21}J_m(\mu) - J_n(\lambda)^T R_{21} = ZR_{11}J_m(\mu) - ZJ_n(\lambda)ZZR_{11}$$

$$\text{therefore } R_{21} = ZR_{11} = \begin{cases} 0 & \text{if } \lambda \neq \mu, \\ R^{\searrow}_{nm} & \text{if } \lambda = \mu. \end{cases}$$

$$(2,1)\text{-block:} \qquad -R_{22}J_m(\mu)^T + J_n(\lambda)^T R_{22} = ZR_{11}ZZJ_m(\mu)Z - ZJ_n(\lambda)ZZR_{11}Z$$

$$\text{therefore } R_{22} = -ZR_{11}Z = \begin{cases} 0 & \text{if } \lambda \neq \mu, \\ S^{\swarrow}_{nm} & \text{if } \lambda = \mu. \end{cases}$$

Altogether we have

$$X_3 = R^T = \begin{cases} 0 & \text{if } \lambda \neq \mu, \\ \begin{bmatrix} P^{\swarrow}_{mn} & R^{\searrow}_{mn} \\ Q^{\nwarrow}_{mn} & S^{\nearrow}_{mn} \end{bmatrix} & \text{if } \lambda = \mu, \end{cases} \text{ and } X_2 = S = \begin{cases} 0 & \text{if } \lambda \neq \mu, \\ \begin{bmatrix} -S^{\swarrow}_{nm} & R^{\searrow}_{nm} \\ Q^{\nwarrow}_{nm} & -P^{\nearrow}_{nm} \end{bmatrix} & \text{if } \lambda = \mu. \end{cases}$$

Therefore we have obtained that the interaction between $H_n(\lambda)$ and $H_m(\mu)$ is of the form $(X_3, X_2) = \mathcal{P}(H_n(\lambda), H_m(\mu))$ (12) and similarly $(X_3, X_2) = \mathcal{P}(K_n, K_m)$ (13) when $A_1 = K_n$ and $A_2 = K_m$. Calculating independent parameters we obtain $d_{KK} \coloneqq \dim(\mathrm{inter}(K_n, K_m)) = 4\min(n,m)$ and

$$d_{HH} \coloneqq \dim(\mathrm{inter}(H_n(\lambda), H_m(\mu))) = \begin{cases} 0 & \text{if } \lambda \neq \mu, \\ 4\min(n,m) & \text{if } \lambda = \mu. \end{cases}$$

3.5 Interaction between L_n and L_m

Due to the explanation in the previous subsection, it suffices to consider

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & F_n \\ -F_n^T & 0 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & G_n \\ -G_n^T & 0 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where R and S are the required $(2n+1)$-by-$(2m+1)$ matrices.

After perfoming the matrix multiplications, we have

$$\begin{bmatrix} -R_{12}F_m^T + F_nS_{21} & R_{11}F_m + F_nS_{22} \\ -R_{22}F_m^T - F_n^TS_{11} & R_{21}F_m - F_n^TS_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\begin{bmatrix} -R_{12}G_m^T + G_nS_{21} & R_{11}G_m + G_nS_{22} \\ -R_{22}G_m^T - G_n^TS_{11} & R_{21}G_m - G_n^TS_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This system of matrix equations decomposes into four independent systems of matrix equations each of them corresponds to one pair of blocks. Consider first the $(1,1)$-blocks:

$$\begin{aligned} -R_{12}F_m^T + F_nS_{21} &= 0, \\ -R_{12}G_m^T + G_nS_{21} &= 0. \end{aligned} \tag{33}$$

From the first equation of (33) we obtain that

$$S_{21} = \begin{bmatrix} & W & \\ a_1 & \dots & a_m \end{bmatrix}, R_{12} = \begin{bmatrix} & & b_1 \\ W & & \vdots \\ & & b_n \end{bmatrix}, \text{ where } W \text{ is any } n\text{-by-}m \text{ matrix.} \tag{34}$$

Therefore

$$-\begin{bmatrix} & & b_1 \\ W & & \vdots \\ & & b_n \end{bmatrix} G_m^T + G_n \begin{bmatrix} & W & \\ a_1 & \dots & a_m \end{bmatrix} = 0,$$

or equivalently

$$\begin{bmatrix} w_{21} - w_{12} & w_{22} - w_{13} & \dots & w_{2m-1} - w_{1m} & w_{2m} - b_1 \\ w_{31} - w_{22} & w_{32} - w_{23} & \dots & w_{3m-1} - w_{2m} & w_{3m} - b_2 \\ \dots & \dots & \dots & \dots & \dots \\ w_{n1} - w_{n-12} & w_{n2} - w_{n-13} & \dots & w_{nm-1} - w_{n-1m} & w_{nm} - b_{m-1} \\ a_1 - w_{n2} & a_2 - w_{n3} & \dots & a_{n-1} - w_{nm} & a_n - b_m \end{bmatrix} = 0,$$

where $q_{ij}, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m$ are entries of W. Thus we have that $S_{21} = Q_{n+1,m}$, i.e., an $(n+1) \times m$ dense Hankel matrix, $R_{12} = Q_{n,m+1}$, i.e., an $n \times (m+1)$ dense Hankel matrix, with common $n \times m$ part (see (34)), and $a_n = b_m$. It follows that the sets of parametric entries of S_{21} and R_{12} are exactly the same, and therefore they are denoted by the same letter. Calculating the number of independent parameters in the solution we obtain that the dimension of the solution space is equal to $n+m$.

Consider now the system of equations corresponding to the $(1,2)$-blocks:

$$\begin{aligned} R_{11}F_m + F_n S_{22} &= 0, \\ R_{11}G_m + G_n S_{22} &= 0. \end{aligned} \tag{35}$$

From the first equation of (35) we can express S_{22} as

$$S_{22} = \begin{bmatrix} & & 0 \\ & -R_{11} & \vdots \\ & & 0 \\ b_1 & \dots & b_{m+1} \end{bmatrix}, \tag{36}$$

where $R_{11} = [r_{ij}]$ is n-by-m. Substituting S_{22} into the second equation of (35) we obtain

$$R_{11}G_m + G_n \begin{bmatrix} & & 0 \\ & -R_{11} & \vdots \\ & & 0 \\ b_1 & \dots & b_{m+1} \end{bmatrix} = 0,$$

or equivalently

$$\begin{bmatrix} -r_{21} & r_{11} - r_{22} & \dots & r_{1m-1} - r_{2m} & r_{1m} \\ -r_{31} & r_{21} - r_{32} & \dots & r_{2m-1} - r_{3m} & r_{2m} \\ \dots & \dots & \dots & \dots & \dots \\ -r_{n1} & r_{n-11} - r_{n2} & \dots & r_{n-1m-1} - r_{nm} & r_{n-1m} \\ b_1 & r_{n1} + b_2 & \dots & r_{nm-1} + b_m & r_{nm} + b_{m+1} \end{bmatrix} = 0.$$

Thus we have that $R_{11} = P^{\leftrightarrow}_{nm}$ and correspondingly $S_{22} = P^{\leftrightarrow}_{n+1,m+1}$. Recall that the $n \times m$ part, starting form the top-left corner of S_{22} is equal to R_{11} (see (36)). Thus the parametric entries of both matrices are the same. Therefore we have that the dimension of the solution space is equal to $m - n + 1$.

The system of equations corresponding to the $(2,1)$-blocks:

$$\begin{aligned} R_{22}F_m^T + F_n^T S_{11} &= 0, \\ R_{22}G_m^T + G_n^T S_{11} &= 0, \end{aligned} \tag{37}$$

is equal to the previous one up to the transposition and interchanging the roles of n and m. Thus we have that $S_{11} = R^{\updownarrow}_{nm}$ and $R_{22} = R^{\updownarrow}_{n+1,m+1}$. Like in the previous case the $n \times m$ part, starting form the top-left corner, of R_{22} is equal to S_{11}. Thus the parametric entries of both matrices are the same. Therefore we have that the dimension of the solution space is equal to $n - m + 1$.

Consider now the system of equations corresponding to the $(2,2)$-blocks:

$$\begin{aligned} R_{21}F_m - F_n^T S_{12} &= 0, \\ R_{21}G_m - G_n^T S_{12} &= 0. \end{aligned} \tag{38}$$

From the first equation of (38) we have

$$R_{21} = \begin{bmatrix} & Q & \\ 0 & \dots & 0 \end{bmatrix}, \quad S_{12} = \begin{bmatrix} & & 0 \\ Q & & \vdots \\ & & 0 \end{bmatrix}, \quad \text{in which } Q = [q_{ij}] \text{ is any } n\text{-by-}m \text{ matrix.}$$

Substituting S_{12} and R_{21} in the second equation of (38) we have

$$\begin{bmatrix} & Q & \\ 0 & \dots & 0 \end{bmatrix} G_m - G_n^T \begin{bmatrix} & & 0 \\ Q & & \vdots \\ & & 0 \end{bmatrix} = 0,$$

or equivalently

$$\begin{bmatrix} 0 & q_{11} & \dots & q_{1m-1} & q_{1m} \\ -q_{11} & q_{21} - q_{12} & \dots & q_{2m-1} - q_{1m} & q_{2m} \\ \multicolumn{5}{c}{\dotfill} \\ -q_{n-11} & q_{n1} - q_{n-12} & \dots & q_{nm-1} - q_{n-1m} & q_{nm} \\ -q_{n1} & -q_{n2} & \dots & -q_{nm} & 0 \end{bmatrix} = 0.$$

Now it follows that $Q = 0_{n\times m}$ and thus both S_{12} and R_{21} are zero blocks. Altogether we have

$$X_3 = R^T = \begin{bmatrix} P^{\leftrightarrow T}_{nm} & 0_{m,n+1} \\ Q_{m+1,n} & R^{\updownarrow T}_{n+1,m+1} \end{bmatrix} \text{ and } X_2 = S = \begin{bmatrix} R^{\updownarrow}_{nm} & 0_{n,m+1} \\ Q_{n+1,m} & P^{\leftrightarrow}_{n+1,m+1} \end{bmatrix}.$$

Therefore we have obtained that the interaction between L_n and L_m is of the form $(X_3, X_2) = \mathcal{P}(L_n, L_m)$ (14) and by calculating the number of independent parameters we obtain

$$d_{LL} := \dim(\mathrm{inter}(L_n, L_m)) = \begin{cases} 2n + 2 & \text{if } n = m, \\ 2\max(n,m) + 1 & \text{if } n \neq m. \end{cases}$$

3.6 Interaction between $H_n(\lambda)$ and K_m

In this subsection we compute off-diagonal blocks of the solution of the system (1) which correspond to the diagonal blocks $H_n(\lambda)$ and K_m :

$$
\begin{aligned}
R\begin{bmatrix} 0 & J_m(0) \\ -J_m(0)^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\
R\begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} S &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}
\tag{39}
$$

where R, S are the required $2n$-by-$2m$ matrices.

From the first equation we have

$$
S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} 0 & J_m(0) \\ -J_m(0)^T & 0 \end{bmatrix},
$$

or equivalently

$$
\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} -R_{22}J_m(0)^T & R_{21}J_m(0) \\ R_{12}J_m(0)^T & -R_{11}J_m(0) \end{bmatrix}.
$$

We substitute this value of S in the second equation, i.e.

$$
\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} \begin{bmatrix} -R_{22}J_m(0)^T & R_{21}J_m(0) \\ R_{12}J_m(0)^T & -R_{11}J_m(0) \end{bmatrix} = 0,
$$

and we get

$$
\begin{bmatrix} -R_{12} + J_n(\lambda)R_{12}J_m(0)^T & R_{11} - J_n(\lambda)R_{11}J_m(0) \\ -R_{22} + J_n(\lambda)^T R_{22}J_m(0)^T & R_{21} - J_n(\lambda)^T R_{21}J_m(0) \end{bmatrix} = 0. \tag{40}
$$

This matrix equation corresponds to four independent matrix equations and we start to consider the $(1,1)$-block: $R_{12} = J_n(\lambda)R_{12}J_m(0)^T$, where $R_{12} = [r_{ij}]$ is $n \times m$. At entry level we have

$$
\begin{aligned}
-r_{ij} + \lambda r_{ij+1} + r_{i+1j+1} &= 0 \quad \text{if } 1 \leqslant i \leqslant n-1, \quad 1 \leqslant j \leqslant m-1, \\
-r_{ij} + \lambda r_{ij+1} &= 0 \quad \text{if } 1 \leqslant j \leqslant m-1, \quad i = n, \\
-r_{ij} &= 0 \quad \text{if } 1 \leqslant i \leqslant n, \quad j = m.
\end{aligned}
$$

This system has only one solution, the trivial one $R_{12} = 0_{n\times m}$.

Analogously to sections 3.2 and 3.4 we multiply the equations in (40) by Z (25) from right, left, or from both sides. Once again using $Z^2 = I$ and $ZJ(\lambda)Z = J(\lambda)^T$ we obtain the general solution of the remaining three equations:

$$(1,2)\text{-block: } R_{11} - J_n(\lambda)R_{11}J_m(0) = -R_{12}Z + J_n(\lambda)R_{12}ZZJ_m(0)^TZ$$
$$\text{thus } R_{11} = -R_{12}Z = 0_{n\times m}.$$

$$(2,1)\text{-block: } -R_{22} + J_n(\lambda)^TR_{22}J_m(0)^T = -ZR_{12} + ZJ_n(\lambda)ZZR_{12}J_m(0)^T$$
$$\text{thus } R_{22} = ZR_{12} = 0_{n\times m}.$$

$$(2,2)\text{-block: } R_{21} - J_n(\lambda)^TR_{21}J_m(0) = -ZR_{12}Z + ZJ_n(\lambda)ZZR_{12}ZZJ_m(0)^TZ$$
$$\text{thus } R_{21} = -ZR_{12}Z = 0_{n\times m}.$$

Altogether we obtain $X_3 = R^T = 0$ and $X_2 = S = 0$. Thus we have proved that the interaction between $H_n(\lambda)$ and K_m is of the form $(X_3, X_2) = \mathcal{P}(H_n(\lambda), K_m)$ (15) and, in particular, $d_{HK} := \dim(\text{inter}(H_n(\lambda), K_m)) = 0$.

3.7 Interaction between $H_n(\lambda)$ and L_m

In this subsection we compute the off-diagonal blocks of the solution of the system (1) which correspond to the diagonal blocks $H_n(\lambda)$ and L_m :

$$R\begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$R\begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where R, S are the required $2n$-by-$(2m+1)$ matrices.

From the first equation we have

$$S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}\begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} -R_{22}F_m^T & R_{21}F_m \\ R_{12}F_m^T & -R_{11}F_m \end{bmatrix}.$$

Substituting this value of S into the second equation we obtain

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} \begin{bmatrix} -R_{22}F_m^T & R_{21}F_m \\ R_{12}F_m^T & -R_{11}F_m \end{bmatrix} = 0,$$

which corresponds to the four independent matrix equations:

$$\begin{bmatrix} -R_{12}G_m^T + J_n(\lambda)R_{12}F_m^T & R_{11}G_m - J_n(\lambda)R_{11}F_m \\ -R_{22}G_m^T + J_n(\lambda)^T R_{22}F_m^T & R_{21}G_m - J_n(\lambda)^T R_{21}F_m \end{bmatrix} = 0.$$

We start to consider the $(1,1)$-block: $R_{12}G_m^T = J_n(\lambda)R_{12}F_m^T$, where the matrix $R_{12} = [r_{ij}]$ is $n \times (m+1)$. At entry level we get

$$\begin{bmatrix} -r_{12} + \lambda r_{11} + r_{21} & -r_{13} + \lambda r_{12} + r_{22} & \dots & -r_{1m+1} + \lambda r_{1m} + r_{2m} \\ -r_{22} + \lambda r_{21} + r_{31} & -r_{23} + \lambda r_{22} + r_{32} & \dots & -r_{2m+1} + \lambda r_{2m} + r_{3m} \\ \dots & \dots & \dots & \dots \\ -r_{n-12} + \lambda r_{n-11} + r_{n1} & -r_{n-13} + \lambda r_{n-12} + r_{n2} & \dots & -r_{n-1m+1} + \lambda r_{n-1m} + r_{nm} \\ -r_{n2} + \lambda r_{n1} & -r_{n3} + \lambda r_{n2} & \dots & -r_{nm+1} + \lambda r_{nm} \end{bmatrix} = 0.$$

If $\lambda = 0$ then the general solution of the system is $R_{12} = P^{\Nwarrow}_{n,m+1}$. If $\lambda \neq 0$ then the answer is more complicated and can be computed recurrently as follows:

$$R_{12} = \begin{cases} r_{i,m+1} = \alpha_{n-i+1}, & 1 \leqslant i \leqslant n \text{ (last column)}, \\ r_{nj} = \frac{\alpha_1}{\lambda^{m-j+1}}, & 1 \leqslant j \leqslant m+1 \text{ (last row)}, \\ r_{ij} = \frac{1}{\lambda}(r_{ij+1} - r_{i+1j}), & 1 \leqslant i < n \text{ and } 1 \leqslant j < m+1. \end{cases}$$

Therefore we have $R_{12}^T = P^{\mathrm{lc}}_{m+1,n}$.

To find the solution for $R_{22}G_m^T = J_n(\lambda)^T R_{22}F_m^T$ we need to multiply the answer for the first equation by Z (see (25)) from the left side because

$$-R_{22}G_m^T + J_n(\lambda)^T R_{22}F_m^T = -ZR_{12}G_m^T + ZJ_n(\lambda)ZZR_{12}F_m^T.$$

Therefore $R_{22} = ZR_{12}$ and thus $R_{22}^T = R_{12}^T Z$ thus for $\lambda = 0$ we have $R_{22} = ZQ^{\Nwarrow}_{n,m+1} = Q^{\Swarrow}_{n,m+1}$ and for $\lambda \neq 0$ we have $R_{22}^T = R_{12}^T Z = P^{\mathrm{lc}}_{m+1,n} Z = P^{\mathrm{fc}}_{m+1,n}$ (in fact, $P^{\mathrm{fc}}_{m+1,n}$ is obtained from $P^{\mathrm{lc}}_{m+1,n}$ by reversing the order of columns).

Consider the equation $R_{11}G_m = J_n(\lambda)R_{11}F_m$, in which $n \times m$ matrix $R_{11} = [r_{ij}]$. Multiplying the matrices we have

$$\begin{bmatrix} -\lambda r_{11} - r_{21} & r_{11} - \lambda r_{12} - r_{22} & \dots & r_{1m-1} - \lambda r_{1m} - r_{2m} & r_{1m} \\ -\lambda r_{21} - r_{31} & r_{21} - \lambda r_{22} - r_{32} & \dots & r_{2m-1} - \lambda r_{2m} - r_{3m} & r_{2m} \\ \dots & \dots & \dots & \dots & \dots \\ -\lambda r_{n-11} - r_{n1} & r_{n-11} - \lambda r_{n-12} - r_{n2} & \dots & r_{n-1m-1} - \lambda r_{n-1m} - r_{nm} & r_{n-1m} \\ -\lambda r_{n1} & r_{n1} - \lambda r_{n2} & \dots & r_{nm-1} - \lambda r_{nm} & r_{nm} \end{bmatrix} = 0.$$

The system has only one solution which is $R_{11} = 0$. Tacking into account that $R_{21}G_m - J_n(\lambda)^T R_{21}F_m = ZR_{11}G_m - ZJ_n(\lambda)ZZR_{11}F_m$ we have $R_{21} = ZR_{11} = 0$. Altogether we get

$$X_3 = R^T = \begin{cases} \begin{bmatrix} 0_{mn} & 0_{mn} \\ P^{\nwarrow T}_{n,m+1} & Q^{\swarrow T}_{n,m+1} \end{bmatrix} & \text{if } \lambda = 0, \\ \begin{bmatrix} 0_{mn} & 0_{mn} \\ P^{\mathrm{lc}}_{m+1,n}(\lambda) & Q^{\mathrm{fc}}_{m+1,n}(\lambda) \end{bmatrix} & \text{if } \lambda \neq 0, \end{cases}$$

$$X_2 = S = \begin{cases} \begin{bmatrix} -Q^{\swarrow}_{n,m+1}F_m^T & 0_{n,m+1} \\ P^{\nwarrow}_{n,m+1}F_m^T & 0_{n,m+1} \end{bmatrix} & \text{if } \lambda = 0, \\ \begin{bmatrix} -(F_m Q^{\mathrm{fc}}_{m+1,n}(\lambda))^T & 0_{n,m+1} \\ (F_m P^{\mathrm{lc}}_{m+1,n}(\lambda))^T & 0_{n,m+1} \end{bmatrix} & \text{if } \lambda \neq 0. \end{cases}$$

Therefore we have obtained that the interaction between $H_n(\lambda)$ and L_m is of the form $(X_3, X_2) = \mathcal{P}(H_n(\lambda), L_m)$ (16) and calculating the number of independent parameters we obtain that $d_{HL} := \dim(\mathrm{inter}(H_n(\lambda), L_m)) = 2n$.

3.8 Interaction between K_n and L_m

In this subsection we compute off-diagonal blocks of the solution of the system (1) which correspond to the diagonal blocks K_n and L_m :

$$R\begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(0) \\ -J_n(0)^T & 0 \end{bmatrix} S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$R\begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where R, S are the required $2n$-by-$(2m+1)$ matrices.

From the second equation we have

$$S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} -R_{22}G_m^T & R_{21}G_m \\ R_{12}G_m^T & -R_{11}G_m \end{bmatrix}.$$

By substituting this value of S in the first equation, i.e.

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(0) \\ -J_n(0)^T & 0 \end{bmatrix} \begin{bmatrix} -R_{22}G_m^T & R_{21}G_m \\ R_{12}G_m^T & -R_{11}G_m \end{bmatrix} = 0,$$

we obtain

$$\begin{bmatrix} -R_{12}F_m^T + J_n(0)R_{12}G_m^T & R_{11}F_m - J_n(0)R_{11}G_m \\ -R_{22}F_m^T + J_n(0)^T R_{22}G_m^T & R_{21}F_m - J_n(0)^T R_{21}G_m \end{bmatrix} = 0. \tag{41}$$

This matrix equation decomposes into the four independent matrix equations each of them corresponds to the one block. We start to consider the $(1,1)$-block: $R_{12}F_m^T = J_n(0)R_{12}G_m^T$, where $R_{12} = [r_{ij}]$ is $n \times (m+1)$. At entry level we have

$$\begin{bmatrix} -r_{11} + r_{22} & -r_{12} + r_{23} & \dots & -r_{1m} + r_{2m+1} \\ -r_{21} + r_{32} & -r_{22} + r_{33} & \dots & -r_{2m} + r_{3m+1} \\ \dots & \dots & \dots & \dots \\ -r_{n-11} + r_{n2} & -r_{n-12} + r_{n3} & \dots & -r_{n-1m} + r_{nm+1} \\ -r_{n1} & -r_{n2} & \dots & -r_{nm} \end{bmatrix} = 0.$$

The solution of the $(1,1)$-block system is $R_{12} = P^{\nearrow}_{n,m+1}$, which also defines S_{21}. Next we consider the $(2,1)$-block of (41). Since,

$$-R_{22}F_m^T + J_n(0)^T R_{22}G_m^T = -ZR_{12}F_m^T + ZJ_n(0)ZZR_{12}G_m^T,$$

we get $R_{22} = ZR_{12} = ZQ^{\nearrow}_{n,m+1} = Q^{\searrow}_{n,m+1}$, where Z is the flip matrix (25).

The $(1,2)$-block of (41) corresponds to $R_{11}F_m = J_n(0)R_{11}G_m$, where matrix $R_{11} = [r_{ij}]$ is $n \times m$. At the entry level we have

$$\begin{bmatrix} r_{11} & r_{12} - r_{21} & r_{13} - r_{22} & \dots & r_{1m} - r_{2m-1} & -r_{2m} \\ r_{21} & r_{22} - r_{31} & r_{13} - r_{32} & \dots & r_{2m} - r_{3m-1} & -r_{2m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ r_{n-11} & r_{n-12} - r_{n1} & r_{n-13} - r_{n2} & \dots & r_{n-1m} - r_{nm-1} & -r_{nm} \\ r_{n1} & r_{n2} & r_{n3} & \dots & r_{nm} & 0 \end{bmatrix} = 0,$$

The system has only one solution, namely $R_{11} = 0_{n\times m}$.

For the $(2,2)$-block we have

$$R_{21}F_m - J_n(0)^T R_{21}G_m = ZR_{11}F_m - ZJ_n(0)ZZR_{11}G_m,$$

and therefore $R_{21} = ZR_{11} = 0$. Altogether we obtain

$$X_3 = R^T = \begin{bmatrix} 0_{mn} & 0_{mn} \\ P^{\nearrow T}_{n,m+1} & Q^{\searrow T}_{n,m+1} \end{bmatrix} \text{ and } X_2 = S = \begin{bmatrix} -Q^{\searrow}_{n,m+1} G_m^T & 0_{n,m+1} \\ P^{\nearrow}_{n,m+1} G_m^T & 0_{n,m+1} \end{bmatrix}.$$

We have proved that the interaction between K_n and L_m is of the form $(X_3, X_2) = \mathcal{P}(K_n, L_m)$ (17) and calculating the number of independent parameters we obtain that $d_{KL} := \dim(\text{inter}(K_n, L_m)) = 2n$.

4 Two examples

We illustrate our results by considering two different pairs of matrix equations (1). Assume that X is the solution of a pair of matrix equations (1) where the matrix pair is taken in canonical form (6). Then the general solution $\widehat{X}$ of (1) can be expressed as $\widehat{X} = SXS^{-1}$, where S is the congruence transformation that brings (A, B) to canonical form.

Example 4.1. Consider the system of matrix equations (1) where $(A, B) = K_1 \oplus K_3 \oplus L_2$ (13×13 skew-symmetric matrix pair). By Theorem 2.1 the solution X of this system is

$$\left(\begin{array}{c|c||ccc|ccc||cc|ccc}
-x_1 & x_2 & -x_7 & 0 & 0 & 0 & 0 & x_6 & 0 & -x_9 & 0 & 0 & 0 \\ \hline
x_3 & x_1 & x_5 & 0 & 0 & 0 & 0 & -x_4 & 0 & x_8 & 0 & 0 & 0 \\ \hline\hline
0 & 0 & -x_{10} & 0 & 0 & 0 & 0 & x_{19} & 0 & -x_{22} & 0 & 0 & 0 \\
0 & 0 & -x_{11} & -x_{10} & 0 & 0 & x_{19} & x_{20} & -x_{22} & -x_{23} & 0 & 0 & 0 \\
x_4 & x_6 & -x_{12} & -x_{11} & -x_{10} & x_{19} & x_{20} & x_{21} & -x_{23} & -x_{24} & 0 & 0 & 0 \\ \hline
x_5 & x_7 & x_{13} & x_{14} & x_{15} & x_{10} & x_{11} & x_{12} & x_{17} & x_{18} & 0 & 0 & 0 \\
0 & 0 & x_{14} & x_{15} & 0 & 0 & x_{10} & x_{11} & x_{16} & x_{17} & 0 & 0 & 0 \\
0 & 0 & x_{15} & 0 & 0 & 0 & 0 & x_{10} & 0 & x_{16} & 0 & 0 & 0 \\ \hline\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_{25} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_{25} & 0 & 0 & 0 \\ \hline
0 & 0 & x_{16} & 0 & 0 & 0 & 0 & x_{22} & x_{26} & x_{27} & x_{25} & 0 & 0 \\
0 & 0 & x_{17} & x_{16} & 0 & 0 & x_{22} & x_{23} & x_{27} & x_{28} & 0 & x_{25} & 0 \\
x_8 & x_9 & x_{18} & x_{17} & x_{16} & x_{22} & x_{23} & x_{24} & x_{28} & x_{29} & 0 & 0 & x_{25}
\end{array}\right),$$

where $x_i \in \mathbb{C}, i = 1, \ldots, 29$.

The dimension of the solution space is $d_{(A,B)} = 29$ (the number of independent parameters in X) and therefore the codimension of the congruence orbit of (A, B) is equal to 16 ($= d_{(A,B)} - n$).

Example 4.2. Consider the system of matrix equations (1) where $(A, B) = H_2(\lambda) \oplus L_2, \lambda \neq 0$ (9×9 skew-symmetric matrix pair). By Theorem 2.1 the solution X of this system is

$$\left(\begin{array}{cc|cc||cc|ccc}
-x_1 & 0 & 0 & x_5 & -\frac{x_9}{\lambda^2} & -\frac{x_9}{\lambda} & 0 & 0 & 0 \\
-x_2 & -x_1 & x_5 & x_6 & -\frac{x_{10}}{\lambda^2} + \frac{2x_9}{\lambda^3} & -\frac{x_{10}}{\lambda} + \frac{x_9}{\lambda^2} & 0 & 0 & 0 \\
\hline
x_3 & x_4 & x_1 & x_2 & \frac{x_8}{\lambda^2} - \frac{2x_7}{\lambda^3} & \frac{x_8}{\lambda} - \frac{x_7}{\lambda^2} & 0 & 0 & 0 \\
x_4 & 0 & 0 & x_1 & \frac{x_7}{\lambda^2} & \frac{x_7}{\lambda} & 0 & 0 & 0 \\
\hline\hline
0 & 0 & 0 & 0 & -x_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -x_{11} & 0 & 0 & 0 \\
\hline
\frac{x_8}{\lambda^2} - \frac{2x_7}{\lambda^3} & \frac{x_7}{\lambda^2} & \frac{x_9}{\lambda^2} & \frac{x_{10}}{\lambda^2} - \frac{2x_9}{\lambda^3} & x_{12} & x_{13} & x_{11} & 0 & 0 \\
\frac{x_8}{\lambda} - \frac{x_7}{\lambda^2} & \frac{x_7}{\lambda} & \frac{x_9}{\lambda} & \frac{x_{10}}{\lambda} - \frac{x_9}{\lambda^2} & x_{13} & x_{14} & 0 & x_{11} & 0 \\
x_8 & x_7 & x_9 & x_{10} & x_{14} & x_{15} & 0 & 0 & x_{11}
\end{array}\right),$$

where $x_i \in \mathbb{C}, i = 1, \ldots, 15$.

The dimension of the solution space is $d_{(A,B)} = 15$ (the number of independent parameters in X) and therefore the codimension of the congruence orbit of (A, B) is equal to 6 $(= d_{(A,B)} - n)$.

Acknowledgements

This work presents research results supported by the Swedish Research Council (VR) under grant A0581501, and by eSSENCE, a strategic collaborative e-Science programme funded by the Swedish Research Council.

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