StratiGraph Tool: Matrix Stratifications in Control Applications

Bo Kågström^{*}, Stefan Johansson[†], and Pedher Johansson[†]

Abstract

In this contribution, the software tool StratiGraph for computing and visualizing closure hierarchy graphs associated with different orbit and bundle stratifications is presented. In addition, we review the underlying theory and illustrate how StratiGraph can be used to analyze descriptor system models via their associated system pencils. The stratification theory provides information for a deeper understanding of how the dynamics of a control system and its system characteristics behave under perturbations.

Key words. Stratification, differential-algebraic equations, descriptor systems, Kronecker structures, orbit, bundle, closure hierarchy, cover relations, StratiGraph.

1 Introduction

Dynamical systems described by linear time-invariant differential-algebraic sets of equations (DAEs) can often be expressed as *descriptor* (or *generalized state-space*) models of the following form [5, 46]:

$$\begin{aligned} E\dot{x} &= Ax + Bu, \\ y &= Cx + Du. \end{aligned} \tag{1}$$

where $E, A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{n \times m}, C \in \mathbb{C}^{p \times n}$, and $D \in \mathbb{C}^{p \times m}$. Determining the system characteristics (like poles, zeros, controllability, and observability) of a model (1) involves computing the canonical structure information of an associated system pencil. In the general case, the *system pencil* has the form

$$\mathbf{S}(\lambda) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \lambda \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix},\tag{2}$$

where $\mathbf{S}(\lambda)$ is of size $(n + p) \times (n + m)$. Such information, including the dynamics of the DAE system $E\dot{x} = Ax + Bu$, is revealed by various canonical forms of the complete system pencil or parts of $\mathbf{S}(\lambda)$. All these features are ill-posed problems in the sense that small perturbations of the matrices (E, A, B, C, D) defining the model can drastically change the system characteristics. The stratification theory for orbits and bundles of matrices, matrix pencils and various system pencils provides information for a deeper understanding of how such system transitions take place under perturbations of the involved matrices.

We start by considering a general matrix pencil $G - \lambda H$ of size 2×4 in order to introduce and illustrate some concepts and results. In Figure 1, closure hierarchy (or

^{*}Department of Computing Science and HPC2N, Umeå University, Sweden. (bokg@cs.umu.se).

[†]Department of Computing Science, Umeå University, Sweden. ({stefanj, pedher}@cs.umu.se).



Figure 1: Orbit (left graph) and bundle (right graph) stratifications of 2×4 matrix pencils computed by StratiGraph.

stratification) graphs of the orbits and bundles of general 2×4 matrix pencils under equivalence transformations are shown. Each node represents an equivalence orbit (or bundle) with its unique canonical structure and each edge between two nodes corresponds to a closure relation. The topmost node (of full row rank) corresponds to the *most generic* canonical structure while the bottom node (pencil with only zero elements) corresponds to the *most degenerate* canonical structure. Since the pencil is rectangular (2 × 4) it is singular by definition and the structure elements are revealed by the Kronecker canonical form of $G - \lambda H$. In the graphs of Figure 1, L_k , L_k^T and $J_k(\mu_i)$ are right singular, left singular, and Jordan blocks, respectively. A more complete definition of different concepts and canonical attributes appears later in the chapter.

Traversing an orbit (or bundle) closure hierarchy gives information about how structure transitions can take place under perturbations. First, traversing the graph upwards, e.g., moving from one orbit to a more generic orbit, can always be done with arbitrary small perturbations of matrices in the descriptor model. Think about setting some zero elements to very small numbers which will change the canonical structure of the perturbed pencil (at least in finite precision arithmetic). Indeed, adding random noise to all elements of the zero pencil (typically) gives a generic pencil. On the other side, moving from a given orbit to a more degenerate model will in general require much larger perturbations of the matrices involved. For example, in control applications it is important to understand how a system model changes from a controllable one to different uncontrollable system models or how perturbations may change the dynamics of the underlying DAE system.

So what is the difference between the orbit and bundle closure hierarchy graphs in Figure 1? For the orbit case, all eigenvalues are kept fixed—only their elementary divisors (Jordan block sizes) may change—and an eigenvalue may even disappear or a new may appear. For the bundle case, specified eigenvalues may as well coalesce or a multiple eigenvalue can split apart in different eigenvalues (not in this example).

Before we go into any further details and explanations, we outline the content of the rest of this chapter. In Section 2, the concept of stratification and some relevant background theory used in this chapter is introduced. Section 3 presents the software tool StratiGraph for computing and visualizing closure hierarchy graphs associated with different orbit and bundle stratifications. For a given orbit, a stratification graph gives information of nearby canonical forms and associated system models. In Section 4, the stratification theory is applied and we illustrate how StratiGraph can be used to analyze sample DAE systems and in control applications. Finally, Section 5 gives an overview of possible new features and problem setups to StratiGraph. Our related ongoing and planned work, including distance information to more degenerate systems is reviewed and illustrated. For a controllable system, a more degenerate system can be the closest uncontrollable one.

2 Stratification theory—some background

The theory of stratification reveals the qualitative information that the closure hierarchy of orbits and bundles provide (e.g., see [14, 15, 19, 28] and references therein). The closure hierarchy is determined by the closure and cover relations among orbits or bundles, where a cover relation guarantees that two orbits or bundles are nearest neighbours in the closure hierarchy. The orbit, for example, of a matrix pencil $G - \lambda H$ consists of all pencils with the same eigenvalues and the same canonical form as $G - \lambda H$, see Section 2.3. A bundle is the union of all orbits with the same canonical form but with unspecified eigenvalues [1]. Figure 1 illustrates an example of orbit and bundle closure hierarchy graphs, where a cover relation is represented by an edge between two nodes. In a stratification, an orbit can never be covered by a less or equally generic orbit. This implies that structures within the closure hierarchy can be ordered by their dimension (or their codimension).

We continue in Section 2.1 to define integer partitions, which are used as a tool for representing the structural information presented in the subsequent Section 2.2. In Section 2.3 the geometry of the matrix pencil space is considered, and we end by reviewing the stratification rules for matrix pencils in Section 2.4.

2.1 Integer partitions

Edelman, Elmroth and Kågström [15] show how canonical structural information (defined in the next section) can be represented as integer partitions such that the closure relations of the various orbits and bundles are revealed by applying a simple set of rules. Below, we define integer partitions and introduce the combinatorial rules which are used in the rest of this chapter.

An integer partition $\kappa = (\kappa_1, \kappa_2, \kappa_3, ...)$ such that $\kappa_1 \ge \kappa_2 \ge ... \ge 0$ is said to dominate another partition ν , i.e., $\kappa > \nu$ if $\kappa_1 + \kappa_2 + ... + \kappa_i \ge \nu_1 + \nu_2 + ... + \nu_i$ for i = 1, 2, ..., where $\nu \ne \kappa$. Different partitions of an integer can in this way form a dominance ordering. If $\kappa > \nu$ and there is no partition μ such that $\kappa > \mu > \nu$, then κ is said to cover ν . Furthermore, the conjugate partition of κ , $\nu = \operatorname{conj}(\kappa)$, is defined such that ν_j are the number of integers in $\kappa \ge j$.

The integer partitions can also be represented as piles of coins in a table, i.e., an integer partition $\kappa = (\kappa_1, \ldots, \kappa_n)$ can be seen as *n* piles of coins where pile *i* has κ_i coins. The covering relation between two integer partitions can then easily be determined: If an integer partition ν can be obtained from κ by moving one coin in κ one column rightward or one row downward and ν remains monotonic decreasing, then κ covers ν . This defines a minimum rightward coin move [15], see Figure 2. The minimal leftward coin move is defined analogously.



Figure 2: Minimum rightward and leftward coin moves illustrate that $\kappa = (3, 2, 2, 1)$ covers $\nu = (3, 2, 1, 1, 1)$ and $\kappa = (3, 2, 2, 1)$ is covered by $\tau = (3, 3, 1, 1)$.

2.2 Structure information and Kronecker canonical form

The system characteristics of a descriptor system (1) are described by the (canonical) structural elements of an associated system pencil. The structural elements we consider in this chapter are: (1) the right minimal indices $(\epsilon_1, \ldots, \epsilon_{r_0})$, with $\epsilon_1 \ge \epsilon_2 \ge \cdots \ge \epsilon_{r_1} > \epsilon_{r_1+1} = \cdots = \epsilon_{r_0} = 0$; (2) the left minimal indices $(\eta_1, \ldots, \eta_{l_0})$, with $\eta_1 \ge \eta_2 \ge \cdots \ge \eta_{l_1} > \eta_{l_1+1} = \cdots = \eta_{l_0} = 0$; (3) the finite elementary divisors on the form

$$(\lambda - \mu_1)^{h_1^{(1)}}, \dots, (\lambda - \mu_1)^{h_{g_1}^{(1)}}, \dots, (\lambda - \mu_q)^{h_1^{(q)}}, \dots, (\lambda - \mu_q)^{h_{g_q}^{(q)}},$$

with $h_1^{(i)} \ge \cdots \ge h_{g_i}^{(i)} \ge 1$ for each of the *q* distinct finite eigenvalues μ_i , $i = 1, \ldots, q$; (4) the *infinite elementary divisors* on the form $1/\lambda^{s_1}, 1/\lambda^{s_2}, \ldots, 1/\lambda^{s_{g_{\infty}}}$, with $s_1 \ge \cdots \ge s_{g_{\infty}} \ge 1$.

For matrix pencils this structural information is revealed by the *Kronecker canonical* form (KCF) [20]. Any $m_{\rm p} \times n_{\rm p}$ matrix pencil $G - \lambda H$ can be transformed into KCF in terms of an equivalence transformation such that

$$U(G - \lambda H)V^{-1} = \operatorname{diag}(L_{\epsilon_1}, \dots, L_{\epsilon_{r_0}}, J(\mu_1), \dots, J(\mu_q), N_{s_1}, \dots, N_{s_{g_\infty}}, L_{\eta_1}^T, \dots, L_{\eta_{l_0}}^T),$$
(3)

where $J(\mu_i) = \text{diag}(J_{h_1}(\mu_i), \dots, J_{h_{g_i}}(\mu_i)), i = 1, \dots, q.$

The blocks $J_{h_k}(\mu_i)$ are $h_k \times h_k$ Jordan blocks associated with each distinct finite eigenvalue μ_i , where each block corresponds to a *finite elementary divisor* of degree h_k , namely $(\lambda - \mu_i)^{h_k}$. The blocks N_{s_k} are $s_k \times s_k$ Jordan blocks for matrix pencils associated with the eigenvalue at infinity, where each block corresponds to an *infinite elementary divisor* of degree s_k , namely $1/\lambda^{s_k}$. Moreover, g_i is the geometric multiplicity of the finite eigenvalues μ_i and g_∞ is the geometric multiplicity of the infinite eigenvalue. These two types of blocks constitute the regular part of a matrix pencil and are defined by

$$J_{h_k}(\mu_i) \equiv \begin{bmatrix} \mu_i - \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \mu_i - \lambda \end{bmatrix}, \quad \text{and} \quad N_{s_k} \equiv \begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & \ddots & -\lambda \\ & & & 1 \end{bmatrix}.$$

If $m_p \neq n_p$ or det $(G - \lambda H) \equiv 0$ for all $\lambda \in \overline{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}$, then the matrix pencil also includes a *singular part* and we say that the matrix pencil is singular. The singular part of the KCF consists of the r_0 right singular blocks L_{ϵ_k} of size $\epsilon_k \times (\epsilon_k + 1)$, corresponding to the right minimal indices ϵ_k , and the l_0 left singular blocks $L_{\eta_k}^T$ of size $(\eta_k + 1) \times \eta_k$, corresponding to the *left minimal indices* η_k . These blocks are defined by

$$L_{\epsilon_k} \equiv \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix}, \text{ and } L_{\eta_k}^T \equiv \begin{bmatrix} -\lambda & & & \\ 1 & \ddots & \\ & \ddots & -\lambda \\ & & 1 \end{bmatrix}.$$

An L_0 and an L_0^T block are of size 0×1 and 1×0 , respectively, and each of them contributes to a column or row of zeros, respectively, in the matrix representation of the KCF.

In the following, we also use a more compact notation by writing the KCF as a direct sum of blocks:

$$U(G - \lambda H)V^{-1} \equiv \mathbb{L} \oplus \mathbb{L}^T \oplus \mathbb{J}(\mu_1) \oplus \cdots \oplus \mathbb{J}(\mu_q) \oplus \mathbb{N},$$

where

$$\mathbb{L} = \bigoplus_{j=1}^{r_0} L_{\epsilon_j}, \quad \mathbb{L}^T = \bigoplus_{j=1}^{l_0} L_{\eta_j}^T, \quad \mathbb{J}(\mu_i) = \bigoplus_{j=1}^{g_i} J_{h_j}(\mu_i), \quad \text{and} \quad \mathbb{N} = \bigoplus_{j=1}^{g_\infty} N_{s_j}$$

The most robust way of computing the canonical structure information is to use *staircase-type algorithms*, which apply unitary transformations to determine Weyr-type characteristics [32, 42, 44, 9, 10]. For Jordan blocks Weyr and Segre characteristics are closely related. In summary, the canonical structural information can be expressed as the *structure integer partitions*:

- $\mathcal{R} = (r_0, r_1, \dots, r_{\epsilon_1})$ where $r_i = \#L_k$ blocks with $k \ge i$.
- $\mathcal{L} = (l_0, l_1, \dots, l_{\eta_1})$ where $l_i = \# L_k^T$ blocks with $k \ge i$.
- $\mathcal{J}_{\mu_i} = (j_1, j_2, \ldots)$ where $j_t = \#J_k(\mu_i)$ blocks with $k \ge t$. \mathcal{J}_{μ_i} is known as the Weyr characteristics of the finite eigenvalue μ_i . \mathcal{J}_{μ_i} is the conjugate partition of the Segre characteristics $h = (h_1, \ldots, h_{g_i})$ defined by h_k , the exponents of the finite elementary divisors.
- $\mathcal{N} = (n_1, n_2, \ldots)$ where $n_t = \#N_k$ with $k \ge t$. \mathcal{N} is known as the Weyr characteristics of the infinite eigenvalue, where \mathcal{N} is the conjugate partition of the Segre characteristics $s = (s_1, \ldots, s_{g_{\infty}})$ defined by s_k , the exponents of the infinite elementary divisors.

Above we have used the notation $G - \lambda H$ with $\lambda \in \overline{\mathbb{C}}$ for a general matrix pencil. However, from a computational point of view it is more appropriate to consider $\beta G - \alpha H$ for all $(\alpha, \beta) \in \mathbb{C}^2$ where the pair (α, β) is a generalized eigenvalue of $G - \lambda H$. If $\beta \neq 0$, then the pair represents the finite eigenvalue α/β , and if $\alpha \neq 0$ and $\beta = 0$ then (α, β) represents an infinite eigenvalue. This notation also applies to general DAE systems $E\dot{x} = Ax + f$, when we want to compute the eigenvalues of $A - \lambda E$. In this chapter, we mainly consider regular descriptor systems where A and E are square. We also remark that if E is nonsingular, the descriptor system can be transformed into a standard state-space form. However, this type of transformation should only be done if E is a well-conditioned matrix. Otherwise, we should keep the DAE formulation and treat it as a descriptor system.

We are also using the concept of *normal-rank* (e.g., see, [44]), which can be defined as $r = n_{\rm p} - r_0 = m_{\rm p} - l_0$, where r_0 and l_0 are the number of right and left singular blocks, respectively. Therefore, a square regular $n_{\rm p} \times n_{\rm p}$ matrix pencil has full normal-rank (it has no singular blocks).

2.3 Matrix pencil space

An $n \times n$ matrix A can be seen as a point in an n^2 -dimensional (matrix) space, one dimension for each parameter of A. Consequently, the union of all $n \times n$ matrices constitutes the entire matrix space, and an orbit of a matrix is a manifold in the space. Similarly, an $m_p \times n_p$ matrix pencil belongs to a $2m_pn_p$ -dimensional space, and an $(n+m) \times (n+p)$ system pencil (2) belongs to an $(n+m)(n+p) + n^2$ -dimensional space. In the system pencil case, the dimension count is done with the assumption that the zero matrices in the λ -part of $\mathbf{S}(\lambda)$ are fixed, while E (typically singular) is not.

In the following, we use a general matrix pencil to illustrate the concepts of orbit, bundle, and their (co)dimensions. The *orbit* of an $m_{\rm p} \times n_{\rm p}$ matrix pencils is the manifold of equivalent matrix pencils:

$$\mathcal{O}(G - \lambda H) = \{ U(G - \lambda H)V^{-1} : \det(U) \cdot \det(V) \neq 0 \}.$$

A bundle defines the union of all orbits with the same canonical form but with the eigenvalues unspecified, $\bigcup_{\mu_i} \mathcal{O}(G - \lambda H)$ [1]. We denote the bundle of $G - \lambda H$ by $\mathcal{B}(G - \lambda H)$.

The dimension of $\mathcal{O}(G - \lambda H)$ is equal to the dimension of the tangent space to $\mathcal{O}(G - \lambda H)$ at $G - \lambda H$, which can be expressed by the pencils on the form

$$T_G - \lambda T_H = X(G - \lambda H) - (G - \lambda H)Y,$$

where X is an $m_{\rm p} \times m_{\rm p}$ matrix and Y is an $n_{\rm p} \times n_{\rm p}$ matrix. In practice, it is more convenient to work with the dimension of the *normal space*, which is the orthogonal complement of the tangent space. The dimension of the normal space is called the *codimension* and is uniquely determined by the Kronecker structure [7, 14]:

$$\operatorname{cod}(G - \lambda H) = c_{\operatorname{Right}} + c_{\operatorname{Left}} + c_{\operatorname{Sing}} + c_{\operatorname{Jor}} + c_{\operatorname{Jor},\operatorname{Sing}},\tag{4}$$

where

$$c_{\text{Right}} = \sum_{\epsilon_i > \epsilon_j} (\epsilon_i - \epsilon_j - 1), \qquad c_{\text{Left}} = \sum_{\eta_i > \eta_j} (\eta_i - \eta_j - 1),$$
$$c_{\text{Sing}} = \sum_{\epsilon_i, \eta_j} (\epsilon_i + \eta_j + 2), \qquad c_{\text{Jor}} = \sum_{i=1}^q \sum_{j=1}^{g_i} (2j - 1) h_j^{(i)} + \sum_{j=1}^{g_{\infty}} (2j - 1) s_j,$$

and

$$c_{\text{Jor,Sing}} = (r_0 + l_0) \left(\sum_{i=1}^q \sum_{j=1}^{g_i} h_j^{(i)} + \sum_{j=1}^{g_\infty} s_j \right).$$

Notably, the codimension of an orbit can also be determined, without knowing the KCF, by computing the dimension of the kernel of Z, where Z is a matrix Kronecker product representation of the tangent space [14, 15]. A reliable way is to apply the singular value decomposition to Z.

Under perturbations the dimension of the tangent space of a matrix pencil orbit may change. Perturbations where the dimension decreases (the codimension increases), which corresponds to less generic (more degenerate) cases, are of special interest in applications especially when the impact can be disastrous. For the most generic rectangular pencil the tangent space of the orbit spans the complete space and hence the codimension is zero. For the most degenerate case these dimensions are reversed for the tangent and normal spaces, respectively.

Since a bundle does not have the eigenvalues specified (or fixed), corresponding to one more degree of freedom for each eigenvalue, the tangent space of a bundle spans one extra dimension for each distinct eigenvalue. In conclusion, the codimension of a bundle is equal to the codimension of the corresponding orbit minus the number of distinct fixed eigenvalues [14, 15].

2.4 Cover rules for matrix pencils

The closure decision problem for orbits of general matrix pencils was solved by Pokrzywa [38], see also later reformulations in [6, 2]. The necessary conditions for an orbit or a bundle of two matrix pencils to be closest neighbours (i.e, has a cover relation) were derived in [2, 6, 38], which was later completed with the sufficient conditions in [15]. These results were in [15] also expressed as *stratification rules*, i.e., combinatorial rules acting on the structure integer partitions defined earlier. The stratification theory has further been developed with the stratification rules for matrix pairs (A, B) [19, 28] and most recently for full normal-rank polynomial matrices $P(\lambda) := P_d \lambda^d + \ldots + P_1 \lambda + P_0$ of degree d [29].

Here, we state the theorem of the stratification rules for finding the closest more degenerate (less generic) matrix pencil(s) in the orbit and bundle closure hierarchies, respectively.

Theorem 2.1 [15] Given the structure integer partitions \mathcal{L} , \mathcal{R} , and \mathcal{J}_{μ_i} of $G - \lambda H$, where $\mu_i \in \overline{\mathbb{C}}$, one of the following if-and-only-if rules finds $\widetilde{G} - \lambda \widetilde{H}$ such that **A.** $\mathcal{O}(G - \lambda H)$ covers $\mathcal{O}(\widetilde{G} - \lambda \widetilde{H})$: **B.** $\mathcal{B}(G - \lambda H)$ covers $\mathcal{B}(\widetilde{G} - \lambda \widetilde{H})$:

- (1) Minimum rightward coin move in \mathcal{R} (or \mathcal{L}).
- (2) If the rightmost column in R (or L) is one single coin, move that coin to a new rightmost column of some J_{μi} (which may be empty initially).
- (3) Minimum leftward coin move in any \mathcal{J}_{μ_i} .
- (4) Let k denote the total number of coins in all of the longest (= lowest) rows from all of the J_{μi}. Remove these k coins, add one more coin to the set, and distribute k + 1 coins to r_p, p = 0,...,t and l_q, q = 0,...,k t 1 such that at least all nonzero columns of R and L are given coins.

- (1) Same as rule 1 on the left.
- (2) Same as rule 2 on the left, except it is only allowed to start a new set corresponding to a new eigenvalue (i.e., no appending to nonempty sets).
- (3) Same as rule 3 on the left.
- (4) Same as rule 4 on the left, but apply only if there exists only one set of coins corresponding to one eigenvalue, or if all sets corresponding to each eigenvalue have at least two rows of coins.
- (5) Let any pair of eigenvalues coalesce, i.e., take the union of their sets of coins.

Rules 1 and 2 are not allowed to make coin moves that affect r_0 (or l_0).

For theorems and explicit formulations of the stratification rules for finding covering orbits we refer to [17, 27]. In the examples presented in Section 4, we make explicit references to the rules in the Theorem 2.1 (e.g., A.(3) when refering to the orbit rule number 3) and we also briefly explain differences with rules for matrix pairs and full normal-rank polynomial matrices in the context of each example. The StratiGraph tool is based on the theory and stratification rules discussed above for different types of problem setups.

3 StratiGraph

StratiGraph¹ is a tool for the computation and visualization of closure hierarchy graphs of orbits and bundles of matrices, matrix pencils and various system pencils (e.g., see [25, 26] and further references therein). Over the years, StratiGraph has grown to a flexible and extendable software tool with a broad range of functionality. In this section, we give a short introduction to the current functionality of StratiGraph version 3.0 and the possibility for extensions. For illustration, the orbit stratification of 2×4 matrix pencils in Figure 1 is used.

3.1 StratiGraph user interface



Figure 3: Sample user interface of a StratiGraph window showing the complete closure hierarchy of orbits of 2×4 matrix pencils.

Figure 3 presents the StratiGraph user interface including an example of a closure hierarchy graph. In the left margin of the graph, the different codimensions of the equivalence orbits are listed. The orbit of the most generic structure has codimension 0, and the most

¹StratiGraph is an acronym for *Strati*fication *Graph*.

degenerate structure has codimension 16 (= 2mn). We remark that different orbits can have the same codimension (here two with codimension 6), and may (but not necessarily so) include different orbits in their closure hierarchies. For example, $\mathcal{O}(2L_0 \oplus 2J_1(\mu_1))$ is included in the closure of $\mathcal{O}(2L_0 \oplus J_2(\mu_1))$ but not in the closure of $\mathcal{O}(2L_0 \oplus J_1(\mu_1) \oplus J_1(\mu_2))$.

By right-clicking an edge, information of the structural change between two connected nodes is shown. In the figure, we see an example where a 2×2 Jordan block splits into two 1×1 Jordan blocks corresponding to the same eigenvalue μ_1 when going downwards in the hierarchy (to less generic structures).

In addition to the main window, two dialog windows can be opened that show information on orbit (or bundle) structure relations and a complete list of structures in the stratification, ordered with respect to increasing codimension. In Figure 3 one of these windows are shown, labeled "Covering structures". The information includes the active structure (defined by clicking on a selected node) and the orbits that are covered by and are covering the active orbit, respectively. This feature is especially useful when there are many nodes and edges in the stratification graph.

Number of nodes: 10 Number of edges: 13	Number of nodes: 10 Number of edges: 13	Number of nodes: 10 Number of edges: 13
_0	_ _0	_ _0
⇒2L,	₱ <i>R</i> : 1 1	⇒R: 2 2
_1	1	
$\mathbf{F}_{L_2} \oplus \mathbf{L}_0$	◆ <i>R</i> : 2 0	₽ <i>R</i> : 2 1 1
_3	3	
$OL_1 \oplus L_0 \oplus J_1(\mu_1)$		
-6	6	- 6
$\odot 2L_0 \oplus J_1(\mu_1) \oplus J_1(\mu_2)$		
© 2L ₀ ⊕ J ₂ (µ₁)	● <i>R</i> :00 <i>J</i> (μ₁):2	
_7	7	
$\Phi L_1 \oplus 2L_0 \oplus L_0^T$	• R: 1 0 0 L: 0	© R: 3 1 L: 1
-8	8	- -8
©2L ₀ ⊕2J₁(μ₁)	● R: 0 0 J(μ₁): 1 1	
_9	9	_ 9
©3L ₀ ⊕L ^T	© R: 0 0 0 L: 1	• R: 3 L: 1 1
	_ _11	_ 11
$\odot 3L_0 \oplus L_0^T \oplus J_1(\mu_1)$		$\odot \mathcal{R}$: 3 \mathcal{L} : 1 $\mathcal{J}(\mu_1)$: 1
- 16	_ _ 16	- - 16
©4L ₀ ⊕ 2L ^T 0	● R: 0 0 0 0 L: 0 0	© R: 4 L: 2

Figure 4: A dialog window showing the canonical information of all expanded nodes in the closure hierarchy of Figure 3 in three different notations. In the middle, the representation using Segre characteristics is shown and to the right the corresponding information using Weyr characteristics.

In the preference side-bar under "Notation" of Figure 3, we see that block notation is chosen, i.e., the canonical structures of the orbits are listed as direct sums of blocks in the associated KCF. The notation can be changed by using the representation of the canonical blocks as indices representing Weyr and Segre characteristics, respectively.

The indices representing Segre characteristics are the numbers and the sizes of the different blocks in the KCF, where \mathcal{R} denotes right singular blocks, \mathcal{L} denotes left singular blocks, and $\mathcal{J}(\mu_1)$ denotes Jordan blocks corresponding to eigenvalue μ_1 . For the Weyr characteristics,

the minimal indices are represented as an ordered integer partition (see definitions in Section 2.1). For example, $\mathcal{J}(\mu_1)$: 2, 2, 1 is read from left to right as; 2 Jordan blocks of size 1×1 or larger, 2 blocks of size 2×2 or larger, and 1 block of size 3×3 or larger, i.e., $\mathcal{J}(\mu_1)$ corresponds to $J_2(\mu_1) \oplus J_3(\mu_1)$. Similarly, \mathcal{R} : 2, 2, 1 for right singular blocks corresponds to $L_1 \oplus L_2$.

In Figure 4, the complete list of canonical information of the closure hierarchy in Figure 3 is presented in the different notations provided in StratiGraph. These are obtained from the second dialog window.

3.2 Working with StratiGraph

The normal way to use StratiGraph is via the built-in wizard for specifying the type of input problem to be considered (see Figure 5). This includes a choice between matrices, matrix pencils or matrix pairs, a choice between the stratification for orbits or bundles, and finally the starting canonical structure, which will be presented and appear as the starting node in the stratification graph. In the figure, we also see an option for full normal-rank polynomial matrices that is available as a prototype plug-in (see Section 3.3.1).



Figure 5: Wizard for specifying a starting canonical structure. First, the problem setup is specified, then the choice between orbit and bundle stratifications, and finally the size of the structure. The choices include the most or least generic structure of a specified size or an arbitrary structure for which the blocks are specified in a new window. In the example, the most generic orbit of size 2×4 is specified.

3.2.1 Graph expansion

Given a starting structure and applying the stratification rules reviewed in Section 2, the graph can be expanded both upwards or downwards from each node, making the graph

larger and larger. A symbol on the node indicates if the node can be further expanded or not. The complete stratification can also be rendered. However, the number of nodes grows exponentially with the problem size. Already a modest problem size can result in thousands of nodes and edges. Therefore, the user will be warned if the operation will be very time and memory consuming.

When the graph is expanded, StratiGraph tries to place the nodes to avoid crossing of edges and edges crossing nodes they are not connected to. However, an optimal node placement regarding the least number of crossings is an NP-complete problem, and depending on the problem size we may not be able to afford computing the complete stratification ahead. The algorithm to place nodes is a compromise between computations, optimal node placement, and trying to let already placed nodes stay at a given position. Still the algorithm does not always give a satisfactory result, or we may wish to arrange nodes in a different way. Therefore, StratiGraph provides the functionality to manually move nodes around on each codimension level in the closure hierarchy graph.

3.3 Extendable software

StratiGraph is developed in Java and consists of almost 450 classes and interfaces. Its purpose to compute and visualize stratification graphs, means that we must be able to enter a starting canonical structure, show structural and other information, expand the graph and finally layout, visualize, save, and print it.

From a design point of view, the aim has been to create a flexible, modular and extendable architecture. Most of the features are controlled in a plug-in like system. Even the built-in problem setups like square matrices, and pencils are handled just like any other plug-in would be handled. Modules can be loaded at run-time, and the plug-in manager notifies different components in the software of changes, be it the graphical interface, or new sets of choices.

StratiGraph has basically three types of plug-ins, namely problem setups, designers and extensions which are briefly described below.

3.3.1 Problem setups

The ability to compute and visualize closure hierarchies of canonical structures, represented both as a graph, and in mathematical notation is, of course, one of the basic functionalities of StratiGraph.

Problem setups basically have four tasks: (i) defining the valid parameters for different kind of canonical blocks and structures that can appear for the given problem type; (ii) defining valid input parameters describing the size of the problem, e.g., square matrices only have one input parameter, n, while matrix pencils have two, m and n (see Figure 5); (iii) given a problem size or a list of valid canonical blocks, define a canonical structure object that will become the start node for further expansion of the stratification graph; (iv) implement a set of stratification rules that specifies structural changes between connected nodes in the expansion upwards and downwards.

Presently, StratiGraph has three built-in problem setups, namely for matrices, matrix pencils and matrix pairs. Available is also a prototype plug-in for full normal-rank polynomial matrices.

3.3.2 Designers

Designers control the visualization of nodes and edges. Given a set of graphical properties, like color sets and fonts, and a canonical structure they define how it will be visualized. The

procedure is the same regardless if it is for printing or for the screen.

StratiGraph has a number of built-in designers, for example, three designers for nodes: a round ball shaped representation (used in Figure 10), a node showing canonical information using different representations, and one that shows the canonical structure represented as blocks in a matrix. The latter is most instructive for structures with few eigenvalues.

3.3.3 Extensions

A program extension plug-in is a fully integrated piece of software that adds functionality to StratiGraph. They can be loaded and unloaded at run-time and interacts fully with the internal flow of events while loaded. Even parts of the GUI can be altered.

One extension built-in with StratiGraph is used to control the internal commands, like specifying a new starting structure etc. A prototype of an extension that makes it possible to interact with Matlab² has also been developed. The aim is to be able to compute the canonical structure of an input setup of matrices in Matlab and then use StratiGraph to investigate nearby structures. Interaction with Matlab also makes it possible to compute quantitative information between pairs of canonical structures as further discussed in Section 5.

4 StratiGraph in applications

In this section, we apply the theory of stratification and illustrate how StratiGraph can be used to analyze qualitative information of some control applications.

The first two examples are problem setups that relate to the solution of linear timeinvariant DAE systems $E\dot{x}(t) = Ax(t) + f(t)$. Then follows a study of an electrical circuit descriptor model from [5]. Finally, we briefly review how controllability characteristics of two mechanical system models can be analyzed using stratification theory for matrix pair and full normal-rank polynomial matrix representations [19, 29].

4.1 High nilpotency index system

Consider a DAE system $E\dot{x}(t) = Ax(t) + f(t)$ with the associated matrix pencil $A - \lambda E = N_6$. The pencil $A - \lambda E$ is regular and has only infinite eigenvalues (its KCF has one block, namely N_6). Indeed, the *(nilpotency) index* of the DAE system is $n_{ind} = 6$ and the DAE system has only an algebraic part (A-part) which requires that f(t) is sufficiently differentiable (at least $n_{ind} - 1$ times). Similar differentiability of f(t) is also a requirement for admissible initial conditions of a general DAE system with regular $A - \lambda E$.

Since a square regular matrix pencil is a first order (d = 1) square polynomial matrix of full normal-rank, we can compute the stratification using the derived rules in [29]. These rules coincide with the rules in Theorem 2.1, with the restrictions that no singular blocks can exist and there can at most exist n Jordan blocks for each eigenvalue (including infinity).

The orbit closure hierarchy graph computed by StratiGraph is shown in Figure 6. By considering the orbit stratification of $A - \lambda E \equiv N_6$ we only allow perturbations that preserve the regularity and keep all eigenvalues at infinity. Starting at $\mathcal{O}(N_6)$ of codimension 6 (= n), the Segre (and Weyr) characteristics of the computed canonical structures in the stratification correspond to the dominance (and reversed dominance) ordering of the integer 6. So $\mathcal{O}(6N_1)$ of codimension 36 (= n^2) corresponds to the most degenerate canonical structure (E is the zero matrix). For example, to obtain the Weyr characteristics, only minimal leftward coin

²Matlab is a registered trademark of The MathWorks, Inc.



Figure 6: The orbit closure hierarchy graph of N_6 , a 6×6 matrix pencil of full normal-rank.

moves (rule A.(3) in Theorem 2.1) are applied to the sequence (1, 1, 1, 1, 1, 1) = conj(6). In addition, the DAE systems corresponding to the pencils in the orbit stratification have their index equal to the size of the largest N_k block in its KCF. Notably, by transversing the closure hierarchy graph downwards we pass (more degenerate) orbits of increasing codimensions while the associated DAE systems have decreasing index. Since all degenerate orbits in the graph belong to the closure of $\mathcal{O}(N_6)$, these cases can be made more generic with arbitrary small perturbations. This is in general not the case for moving downwards in the closure hierarchy.

4.2 DAE system with singular neighbours

Next we consider a DAE system with both a differential part (D-part) and an A-part. The DAE system is represented by a regular 8×8 pencil $A - \lambda E$ with a known KCF $2J_1(\mu) \oplus 2N_3$. For this study we make no restrictions on the perturbations which, for example, means that the stratification may include singular pencils. Already for this small-sized system the complete stratification graph contains 1247 nodes and 4015 edges, which is not easily analyzed. In many cases, it is enough to be able to obtain information about neighboring orbits, and here we demonstrate how the stratification rules are used to compute the orbits in the cover of $\mathcal{O}(A - \lambda E)$.

Figure 7 shows the orbit of $A - \lambda E$ with its nearest neighbours (above and below) in the closure hierarchy. The active node is $\mathcal{O}(2J_1(\mu) \oplus 2N_3)$ with codimension 16. The

orbits downwards in the graph (covered by $\mathcal{O}(A - \lambda E)$) are computed using the A-rules of Theorem 2.1. The orbits upwards in the graph are obtained from the corresponding set of rules for covering orbits (see [15, 27]). For edges going downwards, the corresponding stratification rule and the changes in the KCF are displayed. Actually, only two of the rules are used for $A - \lambda E$. First, rule A.(4) applied to $2J_1(\mu) \oplus 2N_3$ gives four different results: $J_1(\mu) \oplus N_3 \to L_k \oplus L_{3-k}^T$, where $k = \{0, 1, 2, 3\}$. Then rule A.(3) applied to $2J_1(\mu) \oplus 2N_3$ tells us to do a minimum leftward coin move on $\mathcal{N} = (2, 2, 2)$ giving $\mathcal{N} = (3, 2, 1)$, i.e., $2N_3 \to N_3 \oplus N_2 \oplus N_1$. Rules A.(1) and A.(2) cannot be applied since the pencil does not have any singular blocks.



Figure 7: Subgraph of the orbit closure hierarchy of the 8×8 matrix pencil $A - \lambda E = 2J_1(\mu) \oplus 2N_3$ with codimension 16. The A-rules in Theorem 2.1 applied for producing orbits covered by $A - \lambda E$ are marked at two of the edges. Rule A.(4) produces the four singular cases with codimension 17. Rule A.(3) produces the regular case with codimension 18.

We remark that the subgraph in Figure 7 has been generated with a not yet official version of StratiGraph. Current version 3.0 considers all eigenvalues in $\overline{\mathbb{C}}$ similarly, i.e., finite eigenvalues $(J_1(\mu)$ Jordan blocks) and infinite eigenvalues $(N_k$ blocks) are not automatically treated separately.

In Figure 7, the four nodes at codimension level 17 are singular pencils $(\det(A - \lambda E) \equiv 0 \text{ for all } \lambda)$ and the corresponding $E\dot{x}(t) = Ax(t) + f(t)$ are singular systems. Such systems may or may not have a solution, and can even have infinitely many solutions (e.g., see [20, 8] for details).

4.3 Electrical circuit descriptor model

In the following example, we consider the electrical circuit in Figure 8, where the control input u is the voltage source u_e . The resistor, inductor, and capacitors are denoted by R, L,



Figure 8: Electrical circuit from [5].

 C_1 , and C_2 , respectively. Let the state vector be

$$x = \begin{bmatrix} u_{C_1} & u_{C_2} & I_1 & I_2 \end{bmatrix}^T$$

where u_{C_1} and u_{C_2} are the voltages over C_1 and C_2 , respectively, with the corresponding currents I_1 and I_2 . Then according to Kirchoff's second law we obtain a descriptor model of the circuit:

$$E\dot{x} = Ax + Bu$$
, with (5)

$$E = \begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & -L & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & R \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$
(6)

The $n \times (n+m)$ system pencil associated with this descriptor model is 4×5 and has the form

$$\mathbf{S}(\lambda) = \begin{bmatrix} A & B \end{bmatrix} - \lambda \begin{bmatrix} E & 0 \end{bmatrix}.$$
⁽⁷⁾

If R, L, C_1 , and C_2 are nonzero, the system pencil (7) is regular. Moreover, the descriptor circuit model (5) is *completely controllable* since rank $[\lambda E - A, B] = n$ for all $\lambda \in \mathbb{C}$ and rank [E, B] = n [47].

The orbit closure hierarchy graph of a general 4×5 matrix pencil does not take account to any special structure of the system pencil. Instead, we study the system pencil $\mathbf{S}(\lambda)$ in the form of a first order polynomial matrix $P(\lambda) = H\lambda - G$ with $G = \begin{bmatrix} A & B \end{bmatrix}$ and $H = \begin{bmatrix} E & 0 \end{bmatrix}$. $P(\lambda)$ has full normal-rank (= 4) and we can make use of the results in [29] to compute the orbit closure hierarchy graph of a first order full normal-rank polynomial matrix, which is shown in Figure 9. The restrictions of the stratification rules for the full normal-rank polynomial matrices compared to the rules for general matrix pencils are: No left singular blocks (L_k^T) can exist, the number of right singular blocks (L_k) must be m, it can at most exist n Jordan blocks for each eigenvalue (including infinite eigenvalues).

Consider the circuit with the parameters R = 330, L = 1.1, and $C_1 = C_2 = 10^{-4}$. Then $\mathbf{S}(\lambda)$ (7) has the KCF $L_3 \oplus N_1$, is completely controllable, and the pair (E, A) is of index 1 with KCF $3J_1(\alpha) \oplus N_1$. In the graph of Figure 9, this circuit belongs to $\mathcal{O}(L_3 \oplus N_1)$ with codimension 2. Note that we could omit the most generic $\mathcal{O}(L_4)$ in the graph since the matrix E in (6) is always singular. However, in the current version of StratiGraph no such constraints can be specified or imposed automatically.

What happens when we let the parameters R, L, C_1 , and C_2 approach zero? By setting one parameter after another to zero we get canonical structures that belong to less generic orbits further down in the closure hierarchy graph. For example, if we let $L \to 0$ or $C_2 \to 0$ the pencil (7) will approach $\mathcal{O}(L_1 \oplus N_2 \oplus N_1)$ of codimension 8. All transitions are shown in Figure 9. Note that the value of R does not play any role for the computed eigenstructure, i.e., R can be arbitrary. We leave it to the reader to interpret the results from the application's point of view.

A natural question one may ask is: Why do we not get systems that belong to all orbits in the closure hierarchy graph by varying the parameters? The reason is that the matrices in the circuit model (6) are structured with the entries 0 and ± 1 fixed.

4.4 Other examples—two mechanical system models

In this section, we briefly review how the stratification of two mechanical systems can be handled. The first is a linearized model of a uniform platform with two degrees of freedom and the second is a half-car suspension model with four degrees of freedom. These models are



Figure 9: Orbit closure hierarchy graph of a 4×5 full normal-rank polynomial matrix. Transitions obtained by changing the parameters in (6) are shown in the graph with bold arrowed edges (added manually).

analyzed via the state-space model (where $det(E) \neq 0$ in (1)) and via a polynomial matrix representation with a non-singular highest degree coefficient matrix. Only partial results are presented below. For the complete stories we refer to two recent papers [19, 29].

4.4.1 Uniform platform model with two degrees of freedom

First we discuss the stratification of a linearized model of a uniform platform supported in both ends by springs. The linearization of the equations of motion can either be written on the form of a state-space model $\dot{x} = Ax + Bu$, where $A \in \mathbb{C}^{4\times 4}$ and $B \in \mathbb{C}^{4\times 1}$, or as a second order differential equation of the form $M\ddot{x} + C\dot{x} + Kx = Su$, where $M, C, K \in \mathbb{C}^{2\times 2}$, $S \in \mathbb{C}^{2\times 1}$, and $\det(M) \neq 0$. In the most general form, when analyzing the controllability of the platform the stratification can be performed by considering $[A - \lambda I \quad B]$ as a general 4×5 matrix pencil. However, this leads to canonical structures which cannot exist for the platform appearing in the closure hierarchy. In [19], the stratification of the state-space model is studied via the closure hierarchy of the associated 4×5 matrix pair (A, B). In [29], the corresponding study is instead performed on the right linearization

$$\begin{bmatrix} A - \lambda I_4 & B \end{bmatrix} = \begin{bmatrix} \lambda I_2 & M^{-1}K & M^{-1}S \\ -I_2 & \lambda I_2 + M^{-1}C & 0 \end{bmatrix},$$

of the associated 2×3 full normal-rank polynomial matrix to $M\ddot{x} + C\dot{x} + Kx = Su$.

The bundle closure hierarchy graph of a general 4×5 matrix pencil is shown in Figure 10. The light gray area marks the closure hierarchy of a 4×5 matrix pair and the dark gray area marks the closure hierarchy of the right linearization of a 2×3 full normal-rank polynomial matrix. As we see the latter hierarchies form subgraphs of the complete graph and of each other. This follows since the corresponding system pencils have a special predetermined structure which impose restrictions on possible canonical structures.



Figure 10: Bundle closure hierarchy graph of a general 4×5 matrix pencil. The light gray area marks the closure hierarchy of a 4×5 matrix pair (A, B), and the dark gray area the closure hierarchy of the right linearization of a 2×3 full normal-rank polynomial matrix of order two.

The nodes in Figure 10 are shown using the round ball representation labeled with an edge number (at the top) and the codimension of the bundle below. The associated canonical structure information of each node can be obtained in a dialog window (see examples in Figure 4).

4.4.2 Half-car suspension model with four degrees of freedom

A half-car passive suspension model is studied in [29]. The suspension model can be expressed as a fourth-order differential equation

$$P_4 x^{(4)} + P_3 x^{(3)} + P_2 x^{(2)} + P_1 x^{(1)} + P_0 x = Q_2 u^{(2)} + Q_1 u^{(1)} + Q_0 u,$$

where $P_k \in \mathbb{C}^{3\times 3}$, $Q_k \in \mathbb{C}^{3\times 3}$, and $\det(P_4) \neq 0$. The resulting right linearization of the associated 3×6 polynomial matrix is

$$\begin{bmatrix} A - \lambda I_{12} & B \end{bmatrix} = \begin{bmatrix} \lambda I_3 & P_4^{-1}P_0 & P_4^{-1}Q_0 \\ -I_3 & \lambda I_3 & P_4^{-1}P_1 & P_4^{-1}Q_1 \\ & -I_3 & \lambda I_3 & P_4^{-1}P_2 & P_4^{-1}Q_2 \\ & & -I_3 & \lambda I_3 + P_4^{-1}P_3 & 0 \end{bmatrix},$$

where $A \in \mathbb{C}^{12 \times 12}$ and $B \in \mathbb{C}^{12 \times 3}$. The complete stratification of this linearization is rather large. So a possibility when analyzing, for example, the controllability of the suspension model, is to compute the subgraph representing all the controllable orbits (or bundles) together with the closest uncontrollable ones. This subgraph has a moderate size of 35 nodes and 61 edges. For further details we refer to [29], where also the stratification graph is presented.

5 Future work and some open problems

In this section, we discuss possible new features and setups to StratiGraph and related ongoing and planned work as well as some open problems.

One feature is a plug-in for importing and exporting matrix and system pencils between Matlab and StratiGraph. Matlab can also be used as a computing engine to StratiGraph, e.g., to compute quantitative information like bounds on the distance between orbits and bundles in the stratification graph.

Existing problem setups applicable to descriptor systems (1) with a singular E are the matrix pencil setup and in some cases the full normal-rank polynomial matrix setup (currently only available as a prototype). Examples of future setups are general polynomial matrices, descriptor (or singular) system pencils (2), and subpencils corresponding to particular systems.

Version 3.0 of StratiGraph does not separate the finite and infinite eigenvalues when computing and visualizing closure hierarchy graphs. This separation is especially important when we consider general descriptor systems, and will be available in a coming release. A prototype has been implemented in the full normal-rank polynomial matrix setup, as illustrated in Figure 6. In the following, we discuss some of these topics.

5.1 Matrix Canonical Structure Toolbox

For the computational routines and the interaction between Matlab and StratiGraph the *Matrix Canonical Structure* (MCS) toolbox has been developed [24]. The prototype toolbox includes a framework with new data type objects for representing canonical structures and several routines for handling the interface to StratiGraph.

Current version of MCS toolbox includes routines for computing the canonical structure information using staircase algorithms. These are prototype Matlab implementations based on the existing GUPTRI routines [9, 10, 30] for matrix pencils and the controllability and observability staircase forms [43]. Hence, given a linearized model we can compute its Kronecker structure and then let StratiGraph determine and visualize nearby structures in the closure hierarchy. The toolbox also includes prototype implementations of imposed GUPTRI forms, which impose a given canonical structure on an input (matrix, matrix pencil, or matrix pair) with respect to specified deflation tolerances (tol and gap). Other MCS functionality includes routines for computing the distance (measured in the Frobenius norm) between two sets of inputs (matrices, matrix pencils, or matrix pairs), and routines for computing bounds of the distance to a less (or more) generic orbit/bundle [18].

5.2 Bounds and other types of quantitative information

The lower bounds implemented in MCS toolbox are of Eckart-Young type and derived from the matrix representation of the tangent space of the orbit of a matrix or a system pencil [14, 18]. The upper bounds are based on staircase regularizing perturbations [18], which make use of the routines for computing an imposed GUPTRI form. It is well known that these bounds are sometimes too conservative. Future work includes improving the algorithms for computing the bounds for matrices and matrix pencils, especially the upper bound. And it is still open how to compute upper and lower bounds for polynomial matrices and some other particular systems. Another future extension of StratiGraph could be the possibility to compute pseudospectra information (e.g., see [40]) along with distance bounds in the closure hierarchy graph.

Important quantitative information in control applications includes the distances to uncontrollability and unobservability [16, 37]. In [19], a linearized nominal longitudinal model of a Boeing 747 [45] is studied with regard to its controllability characteristics, i.e., the stratification is done on the controllability pair (A, B). The model has 5 states and 5 inputs, and the KCF of (A, B) with the chosen parameters is $2L_2 \oplus L_1 \oplus 2L_0$, i.e., the system is controllable. Using the Matlab routines of Mengi [35], the computed distance to uncontrollability is within the interval $\mathcal{I} = (3.0323 \cdot 10^{-2}, 3.0332 \cdot 10^{-2})$. The results in [19] show that there exist three orbits in the closure hierarchy for which \mathcal{I} is included in the intervals defined by the computed lower and upper bounds. These are the orbits with KCF $L_2 \oplus 2L_1 \oplus 2L_0 \oplus J_1(\mu)$, $2L_2 \oplus 3L_0 \oplus J_1(\mu)$, and $L_3 \oplus L_1 \oplus 3L_0 \oplus J_1(\mu)$, respectively. In Figure 11, a selection of orbits covered by $\mathcal{O}(A,B) \equiv \mathcal{O}(2L_2 \oplus L_1 \oplus 2L_0)$ in the orbit closure hierarchy of 5×10 matrix pairs is shown. We can also see bound information in the figure delivered by the MCS toolbox. For example, the interval \mathcal{I} is within the bounds of $L_2 \oplus 2L_1 \oplus 2L_0 \oplus J_1(\mu_1)$ and $2L_2 \oplus 3L_0 \oplus J_1(\mu_1)$. We remark that these bounds are not as tight as the bounds to the more degenerate uncontrollable system $L_4 \oplus 4L_0 \oplus J_1(\mu_1)$ or the controllable system $L_3 \oplus L_2 \oplus 3L_0$, which has one less input than (A, B) for controlling the states.

5.3 Future problem setups in StratiGraph

Finally, we introduce examples of new problem setups that we aim to include in future versions of StratiGraph. The theory and definitions that are needed to create a problem setup, say for the system pencil $\mathbf{S}(\lambda)$ in (2), are: (1) The orbit of $\mathbf{S}(\lambda)$ must be determined and well-defined (all and only those $\mathbf{S}(\lambda)$ with the same structural elements belong to the same orbit). (2) The codimension of the orbit must be determined and computable from the structural information of $\mathbf{S}(\lambda)$. (3) The closure relation between all possible orbits must be determined and expressed as stratification rules on associated integer partitions.

In general, the stratification theory and covering relations for many of the problem setups listed below are open challenging problems. This also includes definitions of transformations and canonical forms which in turn define the orbits and bundles to be considered. Some related publications include relevant results on stratification theory [22, 23, 4, 12], canonical forms for descriptor systems [21, 34, 41], and staircase and condensed forms for descriptor systems [3, 36, 33].



Figure 11: Subgraph of the orbit closure hierarchy of a 5×10 matrix pair (A, B). The node with double line frame is the active node corresponding to a Boeing 747 under flight $(\mathcal{O}(2L_2 \oplus L_1 \oplus 2L_0))$, and the symbol \bigoplus on the frame indicates unexpanded edges in the graph. The bounds to a selection of orbits are shown together with the eigenvalue of the imposed structure.

5.3.1 State-space descriptor and singular systems

One goal is to cover generalized state-space systems (1) with full generality, including the matrix tuples (E, A, B), (E, A, B, C), and (E, A, B, C, D) where $A - \lambda E$ is regular or even singular. A follow-up extension to these problem setups would be to allow the matrices to be rectangular, which corresponds to various singular state-space systems.

5.3.2 Polynomial matrices

In the case of full normal-rank polynomial matrices, the stratification can be done on the corresponding right linearization [29]. However, for general polynomial matrices there exists, as far as we know, no linearization (canonical form) that preserves the complete eigenstructure. This means that a perturbation in the linearization does not have a one-to-one correspondence with perturbations in the coefficients of the polynomial matrix.

5.3.3 System pencils with structure and other properties

Other problem setups of interest are systems with structure, e.g., fixed elements in the system matrices or with a specified block structure. In addition, it is common that the coefficient matrices of a system model are symmetric, skew-symmetric, Hamiltonian, etc. Many of these applications correspond to matrices and matrix pencils under congruence transformations (e.g., $\mathcal{O}(A) = \{S^T AS : \det(S) \neq 0\}$), [11, 13].

Another challenge is to consider the stratification of generalized matrix products including periodic eigenvalue problems (e.g., see [39, 31]).

Acknowledgements

The authors are grateful to Alan Edelman at MIT, Erik Elmroth at Umeå University, and Paul Van Dooren at UCL Louvain-la-Neuve, co-authors of our fundamental work partly reviewed in this book chapter. We also thank Volker Mehrmann at TU-Berlin and Andras Varga at DLR for suggesting interesting control applications.

This work was supported by the Swedish Foundation for Strategic Research under grant A3 02:128 and UMIT Research Laboratory via an EU Mål 2 project. In addition, support has been provided by eSSENCE, a strategic collaborative e-Science programme funded by the Swedish Research Council.

References

- V. I. ARNOLD, On matrices depending on parameters, Russian Math. Surveys, 26 (1971), pp. 29–43.
- [2] K. BONGARTZ, On degenerations and extensions of finite dimensional modules, Adv. Math., 121 (1996), pp. 245–287.
- [3] A. BUNSE-GERSTNER, R. BYERS, V. MEHRMANN, AND N. K. NICHOLS, Feedback design for regularizing descriptor systems, Linear Algebra Appl., 299 (1999), pp. 119–151.
- [4] J. CLOTET AND M. D. MAGRET, Dimension of orbits of linear time-invariant singular systems under restricted system equivalence, Linear Algebra Appl., 429 (2008), pp. 1102 - 1113.
- [5] L. DAI, Singular Control Systems, vol. 118 of Lecture Notes in Control and Information Science, Springer-Verlag, Berlin, 1989.
- [6] I. DE HOYOS, Points of continuity of the Kronecker canonical form, SIAM J. Matrix Anal. Appl., 11 (1990), pp. 278–300.
- [7] J. DEMMEL AND A. EDELMAN, The dimension of matrices (matrix pencils) with given Jordan (Kronecker) canonical forms, Linear Algebra Appl., 230 (1995), pp. 61–87.
- [8] J. DEMMEL AND B. KÅGSTRÖM, Stably computing the Kronecker structure and reducing subspaces of singular pencils A – λB for uncertain data, in Large Scale Eigenvalue Problems, J. Cullum and R. A. Willoughby, eds., North-Holland, Amsterdam, 1986, pp. 283–323. Mathematics Studies Series Vol. 127, Proceedings of the IBM Institute Workshop on Large Scale Eigenvalue Problems, July 8-12, 1985, Oberlech, Austria.

- [9] —, The generalized Schur decomposition of an arbitrary pencil A λB: Robust software with error bounds and applications. Part I: Theory and algorithms, ACM Trans. Math. Software, 19 (1993), pp. 160–174.
- [10] —, The generalized Schur decomposition of an arbitrary pencil $A \lambda B$: Robust software with error bounds and applications. Part II: Software and applications, ACM Trans. Math. Software, 19 (1993), pp. 175–201.
- [11] F. DE TERÁN, F.M. DOPICO, The solution of the equation $XA + AX^T = 0$ and its application to the theory of orbits, Linear Algebra Appl., 434 (2011), pp. 44–67.
- [12] A. DÍAZ, M. I. GARCÍA-PLANAS, AND S. TARRAGONA, Local perturbations of generalized systems under feedback and derivative feedback, Comput. Math. Appl., 56 (2008), pp. 988–1000.
- [13] A. DMYTRYSHYN, V. FUTORNY, V. V. SERGEICHUK, Miniversal deformations of matrices of bilinear forms, Linear Algebra Appl. 2011 (accepted)
- [14] A. EDELMAN, E. ELMROTH, AND B. KÅGSTRÖM, A geometric approach to perturbation theory of matrices and matrix pencils. Part I: Versal deformations, SIAM J. Matrix Anal. Appl., 18 (1997), pp. 653–692.
- [15] —, A geometric approach to perturbation theory of matrices and matrix pencils. Part II: A stratification-enhanced staircase algorithm, SIAM J. Matrix Anal. Appl., 20 (1999), pp. 667–669.
- [16] R. EISING, Between controllable and uncontrollable, Systems & Control Lett., 4 (1984), pp. 263–264.
- [17] E. ELMROTH, P. JOHANSSON, AND B. KÅGSTRÖM, Computation and presentation of graph displaying closure hierarchies of Jordan and Kronecker structures, Numer. Linear Algebra Appl., 8 (2001), pp. 381–399.
- [18] —, Bounds for the distance between nearby Jordan and Kronecker structures in a closure hierarchy, J. of Mathematical Sciences, 114 (2003), pp. 1765–1779.
- [19] E. ELMROTH, S. JOHANSSON, AND B. KÅGSTRÖM, Stratification of controllability and observability pairs — Theory and use in applications, SIAM J. Matrix Anal. Appl., 31 (2009), pp. 203–226. Also as report UMINF 08.03.
- [20] F. GANTMACHER, The theory of matrices, Vol. I and II (transl.), Chelsea, New York, 1959.
- [21] U. HELMKE AND M. A. SHAYMAN, A canonical form for controllable singular systems, Systems & Control Lett., 12 (1989), pp. 111–122.
- [22] D. HINRICHSEN AND J. O'HALLORAN, A pencil approach to high gain feedback and generalized state space systems, Kybernetika, 31 (1995), pp. 109–139.
- [23] —, Limits of generalized state space systems under proportional and derivative feedback, Math. Control Signals Systems, 10 (1997), pp. 97–124.
- [24] P. JOHANSSON, Matrix Canonical Structure Toolbox, Tech. Rep. UMINF 06.15, Department of Computing Science, Umeå University, Sweden, 2006.

- [25] —, Software tools for matrix canonical computations and web-based software library environments, PhD thesis, Department of Computing Science, Umeå University, Sweden, November 2006.
- [26] —, *StratiGraph homepage*. Department of Computing Science, Umeå University, Sweden, Sept. 2011. http://www.cs.umu.se/english/research/groups/matrix-computations/stratigraph.
- [27] S. JOHANSSON, Reviewing the closure hierarchy of orbits and bundles of system pencils and their canonical forms, Tech. Rep. UMINF 09.02, Department of Computing Science, Umeå University, Sweden, 2009.
- [28] —, Tools for Control System Design Stratification of Matrix Pairs and Periodic Riccati Differential Equation Solvers, Ph.D. Thesis, Report UMINF 09.04, Department of Computing Science, Umeå University, Sweden, March 2009. ISBN 978-91-7264-733-6.
- [29] S. JOHANSSON, B. KÅGSTRÖM, AND P. VAN DOOREN, Stratification of full rank polynomial matrices. Tech. Rep. UMINF 11.11, Department of Computing Science, Umeå University, Sweden, 2011.
- [30] B. KÅGSTRÖM, GUPTRI homepage. Department of Computing Science, Umeå University, Sweden, Sept. 2011. http://www8.cs.umu.se/research/nla/singular_pairs/guptri/.
- [31] B. KÅGSTRÖM, L. KARLSSON, AND D. KRESSNER, Computing Codimensions and Generic Canonical Forms for Generalized Matrix Products, Electronic Journal of Linear Algebra, 22 (2011), pp. 277–309.
- [32] V. N. KUBLANOVSKAYA, On a method of solving the complete eigenvalue problem for a degenerate matrix (in russian), Zh. Vychisl. Mat. Fiz., 6 (1966), pp. 611–620. (USSR Comput. Math. Phys., 6(4):1–16, 1968).
- [33] P. KUNKEL, V. MEHRMANN, AND W. RATH, Analysis and numerical solution of control problems in descriptor form, Math. Control Signals Systems, 14 (2001), pp. 29–61.
- [34] J. J. LOISEAU, K. ÖZQALDIRAN, M. MALABRE, AND N. KARCANIAS, Feedback canonical forms of singular systems, Kybernetika, 27 (1991), pp. 289–305.
- [35] E. MENGI, Software for robust stabilityand controllability measures. Science Department, New York University, Computer NY, Feb. 2008.http://www.cs.nyu.edu/~mengi/robuststability.html.
- [36] P. MISRA, P. VAN DOOREN, AND A. VARGA, Computation of structural invariants of generalized state space systems, Automatica, 30 (1994), pp. 1921–1936.
- [37] C. PAIGE, Properties of numerical algorithms related to computing controllability, IEEE Trans. Autom. Contr., AC-26 (1981), pp. 130–138.
- [38] A. POKRZYWA, On perturbations and the equivalence orbit of a matrix pencil, Linear Algebra Appl., 82 (1986), pp. 99–121.
- [39] V.V. SERGEICHUK, Computation of canonical matrices for chains and cycles of linear mappings, Linear Algebra Appl., 376 (2004), pp. 235–263.
- [40] L. TREFETHEN AND M. EMBREE, Spectra and Pseudospectra: The behavior of nonnormal matrices and operators, Princton University Press, 2005. ISBN 0-691-11946-5.

- [41] D. VAFIADIS AND N. KARCANIAS, Canonical forms for descriptor systems under restricted system equivalence, Automatica, 33 (1997), pp. 955–958.
- [42] P. VAN DOOREN, The computation of Kronecker's canonical form of a singular pencil, Linear Algebra Appl., 27 (1979), pp. 103–141.
- [43] —, The generalized eigenstructure problem in linear system theory, IEEE Trans. Autom. Contr., AC-26 (1981), pp. 111–129.
- [44] —, Reducing subspaces: Definitions, properties and algorithms., in Matrix Pencils, Proc. Pite Havsbad, 1982, B. Kågström and A. Ruhe, eds., vol. 973 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1983, pp. 58–73.
- [45] A. VARGA, On designing least order residual generators for fault detection and isolation, in Proc. of 16th International Conference on Control Systems and Computer Science (CSCS), Bucharest, Rumania, 2007, pp. pp. 323–330.
- [46] G. C. VERGHESE, B. C. LÉVY, AND T. KAILATH, A generalized state-space for singular systems, IEEE Trans. Autom. Contr., AC-26 (1981), pp. 811–830.
- [47] E. YIP AND R. SINCOVEC, Solvability, controllability and observability of continuous descriptor systems, IEEE Trans. Autom. Contr., AC-26 (1981), pp. 702–707.