

Stratification of full rank polynomial matrices

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Dedicated to the memory of Vera Nikolaevna Kublanovskaya —a great personality and source of inspiration [27].

Abstract

We analyze the perturbations of polynomial matrices of full normal-rank via the study of perturbations of linearizations of such polynomial matrices. We show that a full normal-rank polynomial matrix has the same structural elements as its right (or left) linearization. Furthermore, the linearized pencil has a special structure that can be taken into account when studying its stratification. This yields constraints on the set of achievable eigenstructures. We explicitly show which these constrains are. These results allow us to derive necessary and sufficient conditions for cover relations between two orbits or bundles of the linearization of full normal-rank polynomial matrices. The result is exemplified on two mechanical systems: A controlled uniform platform with two degrees of freedom and a half-car passive suspension system with four degrees of freedom.

Keywords: polynomial matrices, matrix pencils, linearization, perturbations, stratification, closure hierarchy, cover relations, StratiGraph

AMS classification: 15A18, 15A21, 15A22, 65F15, 65F35, 93B18

1 Introduction

Polynomial matrices play an important role in the study of dynamical systems described by sets of differential-algebraic equations (DAEs) with constant coefficient matrices

$$P_d x^{(d)}(t) + \ldots + P_1 x^{(1)}(t) + P_0 x(t) = f(t), \ t \ge 0, \tag{1}$$

where $x(\cdot) \in \mathbb{C}^n$, $f(\cdot) \in \mathbb{C}^m$, $P_i \in \mathbb{C}^{m \times n}$, and $x^{(i)}(t)$ is the *i*-th derivative of the vector x(t). Taking the Laplace transform of a DAE system (1) and imposing zero initial conditions, yields the algebraic equation

$$P(s)\hat{x}(s) = \hat{f}(s)$$
 with $P(s) := P_d s^d + \ldots + P_1 s + P_0, s \in \mathbb{C}$,

where d is the degree of P(s), and $\hat{x}(\cdot)$ and $\hat{f}(\cdot)$ are the Laplace transforms of x(t) and f(t), respectively. Throughout the paper, we assume that the leading coefficient matrix P_d is nonzero so that the highest degree is indeed d (we say it has *exact degree d*). The importance of using polynomial models is widely recognized and can be found in basic references such as [17, 36, 26, 34]. For example, polynomial matrices appear when studying linearizations of mechanical systems [37], multibody dynamics [12], and vibration analysis of buildings, machines, and vehicles [29].

When the polynomial matrix P(s) is square and regular (this is when det(P(s)) is not identically zero) then the solutions of the set of differential equations (1) with zero initial conditions mainly depend on the zeros of P(s) and their multiplicities. The fine structure of this so-called zero structure is described in more detail by the elementary divisors of P(s). But if P(s) is singular (this is when det(P(s)) is identically zero for any s or when P(s) is non-square) then the solution set of (1) becomes more complex and depends on the left and right nullspaces of P(s). These null spaces describe, respectively, constraints one needs to impose on f(t) for (1) to have compatible solutions, and degrees of freedom in the solution

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set of (1). It is therefore crucial to understand well the complete eigenstructure of P(s) since this will determine the properties of the solution set of (1).

Unfortunately, the eigenstructure of P(s) is quite sensitive to perturbations in the matrix coefficients P_i and one wants therefore to accurately describe how that structure can change when small variations are applied to these coefficients. Such a study can be performed by so called versal deformations of the eigenstructure of the Jordan and Kronecker canonical forms as introduced in [1] for square matrices. One tool that can be used to analyze the qualitative information of nearby systems is the theory of stratification [10, 9, 13, 25]. A stratification reveals the closure hierarchy of orbits and bundles of nearby canonical structures and gives important qualitative information about the underlying dynamical system. It shows which canonical structures can be represented as a graph where each node represents an orbit or bundle of a canonical structure and an edge corresponds to a covering relation. When two orbits (or bundles) of canonical structures are nearest neighbors in the closure hierarchy they fulfil a cover relation. Such cover relations can be expressed as combinatorial rules acting on integer sequences representing a subset of the structural elements.

Closure and cover relations have been studied, e.g., in [1, 14, 9, 10, 4, 33] for matrices and first order polynomial matrices (matrix pencils), in [13, 16, 19, 21] for system pencils associated with state-space systems, and in [8, 21, 22] for system pencils associated with descriptor and singular systems. In this paper, we extend these results to the case of polynomial matrices by making use of linearizations. These linearizations are matrix pencils, but with the constraint that some elements of the coefficient matrices are 0 or 1, which reduces the set of possible eigenstructures that can be achieved.

Recently problems related to stratification of polynomial matrices have been addressed in [32] and [7]. In [32], it is shown that the map between the orbit space of an controllable matrix pair (A, B) and a polynomial matrix P(s) is a homeomorphism under stated assumptions. The orbits considered are the orbits of matrix pairs under system similarity and the orbits of polynomial matrices under right equivalence. Moreover, necessary and sufficient conditions for a polynomial matrix to be in the closure of another are derived.

The rest of the paper is organized as follows. In Section 2, we start by describing the different eigenstructure elements that a polynomial matrix can have and make the link with the eigenstructure elements of matrix pencils. In Section 3, we describe linearizations of polynomial matrices that preserve these eigenstructure elements. Sections 4 and 5 describe in more detail the *constrained* versal deformations for the so-called scalar and matrix case. In Section 6, we discuss the relations between the polynomial and the pencil representations. We continue in Section 7 to introduce integer partitions and minimal coin moves that are used to represent the structure integer partitions in Section 8 and which appear in the covering rules. Section 9 introduces the polynomial matrix space and we define concepts like orbits and bundles for polynomial matrices, and their codimensions expressed in terms of the structure integer partitions. In Section 10, the cover relations for orbits and bundles of full normal-rank polynomial matrices are derived. Finally, in Section 11 we illustrate and apply the stratification theory on two examples of mechanical systems.

2 Structural elements of P(s)

The eigenstructure elements of a polynomial matrix require the definition of the Smith normal form and of unimodular matrices.

Definition 2.1 A square polynomial matrix M(s) is said to be unimodular if its determinant is constant and non-zero.

Definition 2.2 Two polynomial matrices P(s) and $\tilde{P}(s)$ of the same size are called equivalent if

$$P(s) = M(s)P(s)N(s),$$

for some unimodular matrices M(s) and N(s) of conforming sizes.

Notice that unimodular matrices have a polynomial inverse that is also unimodular and that products of unimodular matrices are also unimodular, from which it follows that they form a transformation group. Under this transformation group a unique canonical form of an arbitrary polynomial matrix can be obtained. **Definition 2.3** [15] The Smith normal form of an arbitrary $m \times n$ polynomial matrix P(s) is the quasi diagonal matrix obtained under unimodular transformations $M_l(s)$ and $M_r(s)$ applied to the rows and columns of P(s):

$$M_{l}(s)P(s)M_{r}(s) = \begin{bmatrix} e_{1}(s) & 0 & \dots & 0 \\ 0 & e_{2}(s) & \ddots & \vdots & & \\ \vdots & \ddots & \ddots & 0 & & \\ 0 & \dots & 0 & e_{r}(s) & & \\ \hline & & & & \\ & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & &$$

where each $e_j(s)$ is monic and divides $e_{j+1}(s)$ for j = 1, ..., r-1. The polynomials $e_j(s)$ are unique and are called the invariant polynomials of P(s).

A (non-unique) zero $\alpha \in \mathbb{C}$ of P(s) is a zero of any $e_j(s)$ and its finite elementary divisors are the factors $(s - \alpha)^{h_j}$ of each $e_j(s)$; their powers are non-increasing:

$$h_1 \ge h_2 \ge \dots \ge h_r \ge 1. \tag{3}$$

The index r is called the normal-rank of P(s) and it is equal to the rank of P(s) at any values of $s \in \mathbb{C}$ which is not a zero of P(s).

We say that an $m \times n$ polynomial matrix has *full normal-rank* if its normal-rank $r = \min(m, n)$. Consequently, if r = m then $n \ge m$ and if r = n then $n \le m$.

For the zero $s = \infty$, there are several different characterizations. We will use here the definition based on the so-called *reversed* polynomial matrix.

Definition 2.4 For a polynomial matrix P(s) of degree d, the reversed polynomial matrix $\operatorname{rev} P(\mu)$ is

$$\operatorname{rev} P(\mu) := \mu^d P(\frac{1}{\mu}) = P_d + P_{d-1}\mu + \ldots + P_0\mu^d,$$
(4)

which is obtained from the substitution $s = \frac{1}{\mu}$ in the polynomial matrix P(s).

Definition 2.5 The finite elementary divisors μ^{h_j} of the zero $\mu = 0$ of $\operatorname{rev} P(\mu)$ are the infinite elementary divisors $1/s^{h_j}$ of the polynomial matrix P(s).

Notice there exist other definitions of infinite zero structure [26, 38] but one can easily find relations between them [39].

A polynomial matrix P(s) that has normal-rank r smaller than m and/or n, has also left and right null spaces that can be represented by polynomial bases as one can see from (2). In order to define the null space structure, we need to define *minimal polynomial bases*.

Definition 2.6 The $n \times r$ polynomial matrix N(s) with the highest column degrees $\{d_1, \ldots, d_r\}$ is column reduced, if the highest degree coefficient matrix N_h , whose *j*-th column is the coefficient of s^{d_j} in the *j*-th column of N(s), also has full column rank. Its normal-rank is therefore also equal to r.

We recall here a lemma about column reduced matrices, that will be useful in the rest of the paper. Proof can be found in, e.g., [26].

Lemma 2.1 Every $n \times r$ polynomial matrix N(s) of normal-rank r can be transformed by a unimodular column transformation V(s) to a column reduced matrix N(s)V(s) with non-increasing column degrees $d_j, j = 1, ..., r$. An additional constant and invertible row transformation R will transform the highest degree coefficient matrix of RN(s)V(s) to $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$.

Remark 2.1 The dual result obviously holds as well. Every $r \times m$ polynomial matrix N(s) of normal-rank r can be transformed by a unimodular row transformation U(s) to a row reduced matrix U(s)N(s) with non-increasing row degrees $d_j, j = 1, \ldots, r$. An additional *constant and invertible* column transformation C will transform the highest degree coefficient matrix of U(s)N(s)C to $\begin{bmatrix} I_r & 0 \end{bmatrix}$.

Definition 2.7 The $n \times r$ polynomial matrix N(s) is called a minimal basis for the space spanned by its columns if N(s) has full column rank for all finite $s \in \mathbb{C}$ and if it is column reduced. The column degrees $\{d_1, \ldots, d_r\}$ of any minimal basis for a particular space, are unique and are called the minimal indices of that space.

We are now ready to define the remaining eigenstructure elements of P(s).

Definition 2.8 Let P(s) be an $m \times n$ polynomial matrix of normal-rank r and let

$$N_{\ell}^{T}(s)P(s) = 0, \quad P(s)N_{r}(s) = 0$$
(5)

where the $m \times (m-r)$ polynomial matrix $N_{\ell}(s)$ and the $n \times (n-r)$ polynomial matrix $N_r(s)$ are column reduced. The left and right null space structures of the polynomial matrix P(s) are then the column degrees $\{\eta_1, \ldots, \eta_{m-r}\}$ and $\{\epsilon_1, \ldots, \epsilon_{n-r}\}$ of $N_{\ell}(s)$ and $N_r(s)$, respectively.

The column degrees $\{\eta_1, \ldots, \eta_{l_0}\}$ and $\{\epsilon_1, \ldots, \epsilon_{r_0}\}$ are called the left (row) and right (column) minimal indices, respectively, where $l_0 = m - r$ and $r_0 = n - r$.

We point out here that if we apply the above definitions to a first order (or linear) polynomial matrix P(s) we retrieve the definitions of the structural elements obtained from the *Kronecker canonical form* (KCF) of a *matrix pencil* sH + G. Any general $m_p \times n_p$ matrix pencil sH + G can be transformed into KCF in terms of an equivalence transformation with two nonsingular matrices U and V [15]:

$$U(sH+G)V^{-1} = \operatorname{diag}(L_{\epsilon_1},\ldots,L_{\epsilon_{r_0}},J(\lambda_1),\ldots,J(\lambda_q),N_{h_1},\ldots,N_{h_{g\infty}},L_{\eta_1}^T,\ldots,L_{\eta_{l_0}}^T),$$
(6)

where $J(\lambda_i) = \text{diag}(J_{h_1}(\lambda_i), \ldots, J_{h_{g_i}}(\lambda_i)), i = 1, \ldots, q$, and g_i is the geometric multiplicity of the finite eigenvalue λ_i and g_{∞} the geometric multiplicity of the infinite eigenvalue. Here λ_i is a distinct eigenvalue of sH + G which coincides with a zero of P(s) in Definition 2.3. The four types of *canonical blocks* are:

$$J_{h_k}(\lambda_i) := \begin{bmatrix} s - \lambda_i & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \\ & & & s - \lambda_i \end{bmatrix}, \qquad N_{h_k} := \begin{bmatrix} -1 & s & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & -1 \end{bmatrix},$$
$$L_{\epsilon_k} := \begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & \ddots & \\ & & & s & -1 \end{bmatrix}, \text{ and } L_{\eta_k}^T := \begin{bmatrix} s & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & \ddots & s \\ & & & & -1 \end{bmatrix},$$

where

- $J_{h_k}(\lambda_i)$ is a $h_k \times h_k$ Jordan block at a finite eigenvalue λ_i , corresponding to a finite elementary divisor of degree h_k , namely $(s \lambda_i)^{h_k}$,
- N_{h_k} is a $h_k \times h_k$ Jordan block at the infinite eigenvalue, corresponding to an infinite elementary divisor of degree h_k , namely $1/s^{h_k}$,
- L_{ϵ_k} is an $\epsilon_k \times (\epsilon_k + 1)$ right singular block, corresponding to a right null vector of minimal degree ϵ_k , namely $[1, s, \ldots, s^{\epsilon_k}]^T$, and
- $L_{\eta_k}^T$ is an $(\eta_k + 1) \times \eta_k$ left singular block, corresponding to a left null vector of minimal degree η_k , namely $[1, s, \ldots, s^{\eta_k}]$.

 L_0 and L_0^T blocks are of size 0×1 and 1×0 , respectively, and each of them contributes with a column or row of zeros in the KCF.

In Section 3, we present a linearization of the polynomial matrix in the form of a matrix pencil and we show that most of the structural elements of P(s) are preserved as the structural elements of the linear pencil.

3 Linearizations

The classical approach to analyze and determine the structural elements of (1) is to study linearizations of polynomial matrices P(s), which result in a large linear matrix pencil sH + G [2, 17]. A linearization is not unique, instead there exist several different, e.g., see [3, 5, 6, 30]. Here we only consider the so called right and left linearizations (also called second and first companion linearizations, respectively). We remark that the companion linearizations are potentially more ill-conditioned relative to P(s). However, when the 2-norms of the coefficient matrices of P(s) are all around one, they are almost equally conditioned [20].

The right linearization of an $m \times n$ polynomial matrix P(s), which is equivalent to the so called second companion form, has the form

$$sH_r + G_r := s \begin{bmatrix} I_m & & & \\ & \ddots & & \\ & & I_m & \\ & & & P_d \end{bmatrix} + \begin{bmatrix} 0 & & P_0 \\ -I_m & \ddots & P_1 \\ & \ddots & 0 & \vdots \\ & & -I_m & P_{d-1} \end{bmatrix}.$$
 (7)

In this section, we show the relations between the eigenstructure elements of P(s) of normal-rank m and those of the matrix pencil $sH_r + G_r$ of size $dm \times dn$. To do this we make use of the following lemma [15].

Lemma 3.1 Two polynomial matrices P(s) and Q(s) are equivalent if and only if they have the same invariant polynomials.

When left multiplying $sH_r + G_r$ (7) with an appropriate unimodular matrix we obtain

$$\begin{bmatrix} I_m & sI_m & \dots & s^{d-1}I_m \\ & \ddots & \ddots & \vdots \\ & & \ddots & sI_m \\ & & & I_m \end{bmatrix} (sH_r + G_r) = \begin{bmatrix} 0 & & P(s) \\ -I_m & \ddots & X_2(s) \\ & \ddots & 0 & \vdots \\ & & -I_m & X_d(s) \end{bmatrix},$$
(8)

where the $X_i(s)$, for i = 2, ..., d, are polynomial matrices as well. An additional unimodular right transformation then gets rid of the matrices $X_i(s)$:

$$\begin{bmatrix} 0 & & P(s) \\ -I_m & \ddots & X_2(s) \\ & \ddots & 0 & \vdots \\ & & -I_m & X_d(s) \end{bmatrix} \begin{bmatrix} I_m & & X_2(s) \\ & \ddots & & \vdots \\ & & I_m & X_d(s) \\ & & & I_m \end{bmatrix} = \begin{bmatrix} & P(s) \\ -I_{m(d-1)} \end{bmatrix}$$

Together with Lemma 3.1 we have now shown that $sH_r + G_r$ and P(s) have the same finite elementary divisors.

For the infinite elementary divisors, we need to compare the elementary divisors of the eigenvalue $\mu = 0$ of the reversed pencil $H_r + \mu G_r$ with those of the reversed polynomial rev $P(\mu)$ defined in (4). We now multiply $H_r + \mu G_r$ on the left with an appropriate unimodular matrix in μ :

$$\begin{bmatrix} I_m & & & \\ \mu I_m & \ddots & & \\ \vdots & \ddots & \ddots & \\ \mu^{d-1} I_m & \dots & \mu I_m & I_m \end{bmatrix} (H_r + \mu G_r) = \begin{bmatrix} I_m & & Y_d(\mu) \\ & \ddots & & \vdots \\ & & I_m & Y_2(\mu) \\ & & & \operatorname{rev} P(\mu) \end{bmatrix}$$

where again the matrices $Y_i(\mu)$ are polynomial matrices in μ and can be eliminated by an additional unimodular transformation applied to the right:

$$\begin{bmatrix} I_m & Y_d(\mu) \\ & \ddots & \vdots \\ & & I_m & Y_2(\mu) \\ & & & \operatorname{rev} P(\mu) \end{bmatrix} \begin{bmatrix} I_m & -Y_d(\mu) \\ & \ddots & \vdots \\ & & I_m & -Y_2(\mu) \\ & & & I_m \end{bmatrix} = \begin{bmatrix} I_{m(d-1)} & \\ & & \operatorname{rev} P(\mu) \end{bmatrix}$$

showing that $H_r + \mu G_r$ and rev $P(\mu)$ have the same elementary divisors. We have thus derived the following theorem (see also [17]).

Theorem 3.2 The polynomial matrix P(s) and the linearized pencil $sH_r + G_r$ defined in (7), have the same finite and infinite elementary divisors.

In order to address the null space structure, we recall a lemma, proved in [39].

Lemma 3.3 Let $\begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = 0$ and let X_1 and $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ have full column rank, then Y_2 must also have full column rank.

We now use this to prove the following theorem for the right null space structures of P(s) and $sH_r + G_r$.

Theorem 3.4 Let N(s) be a minimal basis for the right null space of the pencil $sH_r + G_r$ and partition it as follows (where $N_2(s)$ has n rows):

$$\begin{bmatrix} sI_m & & P_0 \\ -I_m & \ddots & & \vdots \\ & \ddots & sI_m & P_{d-2} \\ & & -I_m & sP_d + P_{d-1} \end{bmatrix} \begin{bmatrix} N_1(s) \\ \hline N_2(s) \end{bmatrix} = 0.$$
(9)

Then $N_2(s)$ is a right minimal basis of P(s) with the same minimal indices as N(s).

Proof. We first apply the same left transformation as in (8) to (9), yielding

$$\begin{bmatrix} 0 & & & P(s) \\ -I_m & \ddots & & & X_2(s) \\ & \ddots & 0 & & \vdots \\ & & & -I_m & X_d(s) \end{bmatrix} \begin{bmatrix} N_1(s) \\ \hline N_2(s) \end{bmatrix} = 0.$$
(10)

Clearly, this implies that $P(s)N_2(s) = 0$ and applying Lemma 3.3 to this for any finite value s, implies that $N_2(s)$ has full column rank for any finite value of s.

Let us now partition the highest degree coefficient matrix N_h in a similar fashion. Then, equating the highest degree coefficients of the top m(d-1) equations of (9) yields

$$\begin{bmatrix} I_m & & 0 \\ & \ddots & \\ & & I_m \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ & & N_{h2} \end{bmatrix} = 0.$$

This implies that $N_{h1} = 0$ and N_{h2} has full column rank. Therefore, $N_2(s)$ is a minimal basis with the same minimal indices as N(s).

Remark 3.1 The following example shows that one can not say the same for the left minimal indices of P(s) and $sH_r + G_r$:

$$P(s) := \begin{bmatrix} s & s^2 \\ 1 & s \end{bmatrix} = \begin{bmatrix} s \\ 1 \end{bmatrix} \begin{bmatrix} 1 & s \end{bmatrix}, \quad sH_r + G_r = \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & s & 1 & 0 \\ \hline -1 & 0 & 1 & s \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

Indeed, the minimal left null spaces of P(s) and $sH_r + G_r$ are respectively $\begin{bmatrix} 1 & -s \end{bmatrix}$ and $\begin{bmatrix} 1 & -s & s & -s^2 \end{bmatrix}$ and their minimal index is different.

Thus we have proved that an $m \times n$ polynomial matrix P(s) of normal-rank m has the same structural elements as the so-called *right* linearization $sH_r + G_r$ (7).

For the left minimal indices we consider the *left* linearization

$$sH_{\ell} + G_{\ell} := s \begin{bmatrix} sI_n & & & \\ & \ddots & & \\ & & sI_n & \\ & & & P_d \end{bmatrix} + \begin{bmatrix} 0 & -I_n & & \\ & \ddots & \ddots & \\ & & 0 & -I_n \\ P_0 & P_1 & \dots & P_{d-1} \end{bmatrix},$$
(11)

for which the dual result holds. Notably, the matrix pencil $sH_{\ell} + G_{\ell}$ is equivalent to the so called *first* companion form.

We synthesize the results of this section in the following theorem.

Theorem 3.5 Let P(s) be an $m \times n$ polynomial matrix of normal-rank r, then

- 1. if r = m, P(s) has the same structural elements as $sH_r + G_r$ defined in (7),
- 2. if r = n, P(s) has the same structural elements as $sH_{\ell} + G_{\ell}$ defined in (11),
- 3. for any r, P(s) has the same elementary divisors as $sH_r + G_r$ and as $sH_\ell + G_\ell$, and
- 4. for any r, P(s) has the same right minimal indices as $sH_r + G_r$ and the same left minimal indices as $sH_\ell + G_\ell$.

4 Scalar case

In this section, we look at the case where m = 1 and we assume that the polynomial matrix has exact degree d (nonzero leading coefficient P_d). This of course implies that the polynomial matrix has normal-rank 1 as well since it is nonzero. The Smith form of such a polynomial matrix is quite special since it contains exactly one polynomial e(s), which is the greatest common divisor of the scalar polynomials in P(s):

$$P(s) := [p_1(s) \dots p_n(s)], \quad e(s) := \gcd \{p_1(s), \dots, p_n(s)\}$$

If k is the degree of e(s) then there are n-2 right minimal indices equal to 0 and one equal to d-k. The other structure elements are all the possible structures one can find in a scalar polynomial of degree k. We synthesize the conclusions in the following theorem.

Theorem 4.1 A $1 \times n$ polynomial matrix P(s) of exact degree d has only one elementary divisor $(s - \lambda_i)^{h_i}$ for each zero λ_i , n-2 right minimal indices equal to zero, and one right minimal index equal to ϵ_1 satisfying

$$\sum_{i} h_i + \epsilon_1 = d.$$

All structures satisfying these constraints are possible for such a polynomial matrix.

Corollary 4.2 A 1×1 scalar polynomial p(s) of exact degree d has only one elementary divisor $(s - \lambda_i)^{h_i}$ for each zero λ_i satisfying

$$\sum_{i} h_i = d.$$

All structures satisfying these constraints are possible for such a polynomial.

Clearly this is not reflected in the general form sH + G of the pencil $sH_r + G_r$, but it is a result of the fact that $sH_r + G_r$ has fixed elements equal to 0 and 1. This problem is also related to the controllability of a generalized state-space system with n - 1 inputs. For this, we related the polynomials as follows:

$$a(s) := p_1(s), \quad B(s) := \begin{bmatrix} p_2(s) & \dots & p_n(s) \end{bmatrix},$$

where we assume for simplicity that the highest degree coefficient of a(s) is non-zero and that of B(s) is equal to zero. This can be achieved by a constant row transformation of P(s) (which does not affect the conclusions), where the highest degree coefficient of a(s) is used as pivot to eliminate those of B(s). One can then consider the following partitioning of the linearization of P(s):

$$\begin{bmatrix} sE + A \mid B \end{bmatrix} := \begin{bmatrix} s & a_0 & B_0 \\ -1 & \ddots & a_1 & B_1 \\ & \ddots & s & \vdots & \vdots \\ & & -1 & sa_d + a_{d-1} & B_{d-1} \end{bmatrix},$$

where B_0, \ldots, B_{d-1} are $1 \times (n-1)$ matrices. The controllability of this generalized state-space pair is equivalent to the existence of a gcd e(s) of the polynomials of P(s) = [a(s) B(s)]. The dimension of the uncontrollable space is also equal to the degree of e(s). Rather than analyzing this using the perturbations of $[sE - A \mid B]$ one can look at perturbations for this row vector of polynomials to have a common divisor.

5 Matrix case

In this section, we derive similar conditions for full normal-rank polynomial matrices as for the scalar case presented in Theorem 4.1 and Corollary 4.2. We will prove the following results.

Theorem 5.1 An $m \times n$ polynomial matrix P(s) of exact degree d and normal-rank m has m finite elementary divisors $(s - \lambda_i)^{h_j^{(i)}}, j = 1, ..., m$, for each zero λ_i , m infinite elementary divisors $1/s^{h_j^{(\infty)}}$, and n-m right minimal indices $\epsilon_j, j = 1, ..., n-m$ (some of these indices can be trivially zero) satisfying

$$\sum_{j=1}^{m} \sum_{i=1}^{q} h_j^{(i)} + \sum_{j=1}^{m} h_j^{(\infty)} + \sum_{j=1}^{n-m} \epsilon_j = dm.$$
(12)

All structures satisfying these constraints are possible for such a polynomial matrix.

Remark 5.1 The dual result for when P(s) has normal-rank n is given by interchanging m and n, and replacing the right minimal indices ϵ with the left minimal indices η in the theorem above.

Corollary 5.2 An $m \times m$ polynomial matrix P(s) of exact degree d and normal-rank m has m finite elementary divisors $(s - \lambda_i)^{h_j^{(i)}}, j = 1, ..., m$, for each zero λ_i , and m infinite elementary divisors $1/s^{h_j^{(\infty)}}$ (some of these indices can be trivially zero) satisfying

$$\sum_{j=1}^{m} \sum_{i=1}^{q} h_j^{(i)} + \sum_{j=1}^{m} h_j^{(\infty)} = dm.$$
(13)

All structures satisfying these constraints are possible for such a polynomial matrix.

We thus need to show that all these structures may occur in an $m \times n$ polynomial matrix P(s) of exact degree d. The fact that these constraints are necessary, is evident since an $m \times n$ polynomial matrix of rank r = m can have only r non-trivial elementary divisors for each zero. The fact that these constraints are sufficient, on the other hand, requires a proof. Our proof is based on unimodular transformations, which leave the finite elementary divisors unchanged, but may change the infinite elementary divisors. We therefore make a change of variables, such that the polynomial matrix has no finite elementary divisors at infinity. For this we need the following lemma.

Lemma 5.3 Let P(s) be an $m \times n$ polynomial matrix of exact degree d and full normal-rank m which has no zero at $s = \omega$. Then putting $s = \frac{1}{\mu} + \omega$, the transformed polynomial matrix

$$P_{\omega}(\mu) := \mu^{d} P(\frac{1}{\mu} + \omega) = P_{d}(1 + \mu\omega)^{d} + P_{d-1}\mu(1 + \mu\omega)^{(d-1)} + \dots + P_{0}\mu^{d}$$

has the same right nullspace structure as P(s), no zero at infinity, and its finite elementary divisors are given by

$$\left(\mu - \frac{1}{(\lambda_i - \omega)}\right)^{h_j^{(i)}}, \quad j = 1, \dots, m, \text{ and } \mu^{h_j^{(\infty)}}, \quad j = 1, \dots, m.$$
 (14)

Proof. This follows directly from the correspondence with the Kronecker structure of the linearized pencil sH + G. The linearization of the transformed polynomial matrix $P_{\omega}(\mu)$ is given by $(1 + \mu\omega)H + \mu G = \mu(\omega H + G) + H$, which has the same right null space structure as sH + G and the same elementary divisors except for the transformations given in (14). \Box

Note that a full normal-rank polynomial matrix P(s) without zeros at infinity must have a highest degree coefficient matrix P_d which has full rank as well. We now show that the result of Corollary 5.2 holds for a polynomial matrix without infinite elementary divisors.

Proof of Corollary 5.2. Let P(s) be in Smith canonical form and $h_1 \ge \cdots \ge h_m \ge 1$ be the degrees of its elementary divisors, then the highest degree coefficient matrix is the identity matrix I_m . Moreover, since P(s) has no elementary divisors at infinity, Corollary 5.2 imposes that $\sum_{i=1}^{m} h_i = dm$. We now show that such a polynomial matrix can be transformed using unimodular transformations to one of degree d with highest degree coefficient matrix $P_d = I_m$. This would imply that there always is a polynomial matrix of degree d that satisfies Corollary 5.2.

We construct such a polynomial matrix by recursively reducing the difference between h_1 and h_m while $\sum_{i=1}^{m} h_i$ remains the same. At the end of this process all h_i will be equal to d. Assume for this

that $h_1 > d$ then we must have $h_m < d$, otherwise the ordered sequence of h_i could not sum up to dm. In the transformation $U(s)P(s)V(s) = \hat{P}(s)$ below, we only show the elements of highest column degrees in P(s) and $\hat{P}(s)$:

$$\begin{bmatrix} & 1 \\ I & \\ 1 & s \end{bmatrix} \begin{bmatrix} s^{h_1} & & \\ & \ddots & \\ & & s^{h_m} \end{bmatrix} \begin{bmatrix} -1 & & \\ & I & \\ s^{\delta} & & 1 \end{bmatrix} = \begin{bmatrix} s^{h_1 - 1} & & \\ & \ddots & \\ xs^{h_1 - 1} & & s^{h_m + 1} \end{bmatrix},$$
 (15)

where $\delta = h_1 - h_m - 1$ and x is arbitrary. Clearly, this transformation yields a new column reduced matrix but with the smallest column degree increased by one and the largest column degree decreased by one. The reduction can be continued since we can use Lemma 2.1 to put again the new matrix in normalized column reduced form. Eventually we obtain an $m \times m$ polynomial matrix of degree d with prescribed elementary divisors and $P_d = I_m$. \Box

Proof of Theorem 5.1. In order to prove Theorem 5.1, we have to construct an $m \times n$ polynomial matrix P(s) with given Smith form, but also with given right minimal indices. In [26], it is shown how to construct a polynomial matrix $P_r(s)$ with prescribed right minimal basis $N_r(s)$, such that $P_r(s)$ has full column rank for all finite $s \in \mathbb{C}$ (i.e. no nontrivial elementary divisors). If we pre-multiply $P_r(s)$ with a diagonal matrix $P_f(s)$ of given finite elementary divisors as shown above, then $P(s) := P_f(s)P_r(s)$ satisfies $P(s)N_r(s) = 0$ and it has the prescribed elementary divisors. According to Remark 2.1 we can further transform it to a row reduced matrix W(s)P(s)R with highest row degree coefficient matrix $P_h = \begin{bmatrix} 0 & I_m \end{bmatrix}$ and with non-increasing row degrees d_1, \ldots, d_m . This new matrix W(s)P(s)R has still the prescribed elementary divisors; the new matrix $R^{-1}N_r(s)$ is a minimal basis for W(s)P(s)R and its highest column degree matrix must be of the type $\begin{bmatrix} C \\ 0 \end{bmatrix}$ since it is orthogonal to P_h . Moreover, C is invertible since $R^{-1}N_r(s)$ is still column reduced. The further transformation $R^{-1}N_r(s)C^{-1}$ still yields a right minimal basis but now with highest degree matrix $N_h = \begin{bmatrix} I_{n-m} \\ 0 \end{bmatrix}$; we will denote the corresponding column degrees by $\epsilon_1, \ldots, \epsilon_{n-m}$ since they are the right minimal indices of P(s).

We now assume that we start with a pair of matrices $P(s)N_r(s) = 0$ with the above conditions on the highest degree coefficient matrices. The conditions of Theorem 5.1 imply that $\sum_{i=1}^{m} d_i = dm$. If all the coefficients d_i are not equal to d we again update the pair.

We now update simultaneously the matrices P(s) and $N_r(s)$ while making sure that: (i) $N_r(s)$ remains a right minimal basis of P(s) with the same minimal indices; and (ii) P(s) has the same invariant polynomials. In the transformation $U(s)P(s)V(s) = \hat{P}(s)$ displayed below, we only show the elements of highest row degrees in P(s) and $\hat{P}(s)$:

$$\begin{bmatrix} -1 & s^{\delta} \\ I & \\ & 1 \end{bmatrix} \begin{bmatrix} 0 & s^{d_1} & \\ & \ddots & \\ & s^{d_m} \end{bmatrix} \begin{bmatrix} I_{n-m} & 0 & \\ & 1 & \\ 0 & I & \\ & 1 & s \end{bmatrix}$$
$$= \begin{bmatrix} 0 & s^{d_1-1} & \\ & \ddots & \\ ss^{d_1-1} & s^{d_m+1} \end{bmatrix},$$

where $\delta = d_1 - d_m - 1$ and x is arbitrary. Just as in the previous reduction these transformations are unimodular and hence the elementary divisors of P(s) and $\hat{P}(s)$ are the same but the smallest row index d_m increased by 1 and the largest row index d_1 decreased by 1. Meanwhile, the right minimal indices did not change, because the corresponding right nullspace underwent the transformation $V^{-1}(s)N_r(s) = \hat{N}_r(s)$. Below we show only the highest column degree elements in $N_r(s)$ and $\hat{N}_r(s)$:

$$\begin{bmatrix} I_{n-m} & 0 \\ & -s & 1 \\ 0 & I & \\ & 1 & - \end{bmatrix} \begin{bmatrix} s^{\epsilon_1} & & & \\ & \ddots & & \\ & s^{\epsilon_{n-m}} \\ & & & \end{bmatrix} = \begin{bmatrix} s^{\epsilon_1} & & & \\ & \ddots & & \\ & s^{\epsilon_{n-m}} \\ \hline & xs^{\epsilon_1} & \cdots & xs^{\epsilon_{n-m}} \\ & & & & \end{bmatrix}.$$

Clearly, only row n + 1 of $\hat{N}_r(s)$ may contribute to the highest degree matrix, but it will not affect the minimal indices. Again, we can continue the recursive updating transformations until all powers $d_i = d$, which completes the proof of Theorem 5.1. \Box

For completeness we include the following two corollaries for polynomial matrices with a full rank highest degree coefficient matrix P_d . The proofs are omitted since, as shown above, a full normal-rank polynomial matrix can always be transformed to one with a full rank P_d via a change of variables.

Corollary 5.4 Let $P(s) = P_d s^d + \ldots + P_1 s + P_0$ be an $m \times n$ polynomial matrix of exact degree d, normalrank m, and with P_d of full row rank. Possible structural elements of P(s) are those of Theorem 5.1 excluding the infinite elementary divisors.

Corollary 5.5 Let $P(s) = P_d s^d + \ldots + P_1 s + P_0$ be an $m \times m$ polynomial matrix of exact degree d, normalrank m, and det $(P_d) \neq 0$. Possible structural elements of P(s) are those of Corollary 5.2 excluding the infinite elementary divisors.

6 Polynomial versus pencil representation

The matrix pencils $sH_r + G_r$ and $sH_\ell + G_\ell$ corresponding to the right and left linearizations of a full normal-rank $m \times n$ polynomial matrix $P(s) = P_d s^d + \ldots + P_1 s + P_0$, can be expressed as the system pencils

$$\mathbf{S}_{\mathrm{R}}(s) = sH_r + G_r = s\begin{bmatrix} E & 0 \end{bmatrix} + \begin{bmatrix} A & B \end{bmatrix} \quad \text{and} \quad \mathbf{S}_{\mathrm{L}}(s) = sH_\ell + G_\ell = s\begin{bmatrix} E \\ 0 \end{bmatrix} + \begin{bmatrix} A \\ C \end{bmatrix}, \tag{16}$$

respectively. If the highest degree coefficient matrix P_d has full row or column rank, the system pencils in (16) can be transformed into

$$\mathbf{S}_{\mathcal{C}}(s) = s \begin{bmatrix} I_{dm} & 0 \end{bmatrix} + \begin{bmatrix} \widetilde{A} & \widetilde{B} \end{bmatrix} \quad \text{or} \quad \mathbf{S}_{\mathcal{O}}(s) = s \begin{bmatrix} I_{dn} \\ 0 \end{bmatrix} + \begin{bmatrix} \widetilde{A} \\ \widetilde{C} \end{bmatrix}, \tag{17}$$

respectively, where $\mathbf{S}_{C}(s)$ has full row rank and $\mathbf{S}_{O}(s)$ has full column rank. The structural elements of $\mathbf{S}_{C}(s)$ only depend on the matrix pair $(\widetilde{A}, \widetilde{B})$ and those of $\mathbf{S}_{O}(s)$ on $(\widetilde{A}, \widetilde{C})$. In the next section, we show that the stratification rules for $\mathbf{S}_{R}(s)$ and $\mathbf{S}_{L}(s)$ can be derived from the stratification rules for general matrix pencils sH + G, and the rules for $\mathbf{S}_{C}(s)$ and $\mathbf{S}_{O}(s)$ from general matrix pairs (A, B) and (A, C), respectively.

In the following, we illustrate how a polynomial matrix can be expressed in the form of a system pencil using three examples. We focus on polynomial matrices of full normal-rank and with a highest degree coefficient matrix P_d of full row rank. The first example is of general form and the remaining two are taken from applications.

Example 6.1 Consider the differential equation

$$D_d x^{(d)}(t) + D_{d-1} x^{(d-1)}(t) + \dots + D_1 x^{(1)}(t) + D_0 x(t) = -N_{d-1} u^{(d-1)}(t) - \dots - N_1 u^{(1)}(t) - N_0 u(t),$$
(18)

where $D_k \in \mathbb{C}^{m \times m}$ and $N_k \in \mathbb{C}^{m \times p}$. It can be expressed as

$$\begin{bmatrix} D_d & 0 \end{bmatrix} \begin{bmatrix} x^{(d)}(t) \\ u^{(d)}(t) \end{bmatrix} + \begin{bmatrix} D_{d-1} & N_{d-1} \end{bmatrix} \begin{bmatrix} x^{(d-1)}(t) \\ u^{(d-1)}(t) \end{bmatrix} + \dots + \begin{bmatrix} D_0 & N_0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = 0,$$
(19)

with the associated $m \times (m+p)$ polynomial matrix $P(s) = \begin{bmatrix} D_d & 0 \end{bmatrix} s^d + \ldots + \begin{bmatrix} D_0 & N_0 \end{bmatrix}$ of degree d. If P(s) has full normal-rank m and $\det(D_d) \neq 0$ then we have the companion linearization

$$s \begin{bmatrix} I_m & & & 0 \\ & \ddots & & & \vdots \\ & & & I_m & 0 \end{bmatrix} + \begin{bmatrix} 0 & & D_d^{-1}D_0 & D_d^{-1}N_0 \\ -I_m & \ddots & D_d^{-1}D_1 & D_d^{-1}N_1 \\ & \ddots & 0 & \vdots & \\ & & -I_m & D_d^{-1}D_{d-1} & D_d^{-1}N_{d-1} \end{bmatrix},$$
(20)

which is a system pencil of the form $\mathbf{S}_{\mathbf{C}}(s)$. Similarly, the left linearization of a polynomial matrix with full normal-rank n and $\det(D_d) \neq 0$ corresponds to the system pencil $\mathbf{S}_{\mathbf{O}}(s)$.

Example **6.2** Consider a controlled dynamical system which can be expressed by its equation of motion on the form

$$M\ddot{x} + C\dot{x} + Kx = Eu,$$

where M, C, and K are the mass, damping, and stiffness matrices, respectively, E is the input (control) matrix, x is a vector of positive variables, and u is a vector of control variables. Assuming the mass matrix M is positive definite, the linearization of the associated polynomial matrix can be expressed by the companion form

$$\begin{bmatrix} sI + A \mid B \end{bmatrix} = \begin{bmatrix} sI & M^{-1}K \\ -I & sI + M^{-1}C \mid 0 \end{bmatrix},$$

where I is the identity matrix of conforming size. The (2,3)-block is a zero matrix since \dot{u} does not appear in the equation of motion.

Example 6.3 Consider an LTI system represented by the state-space model

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{21}$$

where $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times m}$, x(t) is the state vector, and u(t) is the input vector.

The controllability of an LTI system only depends on the matrices A and B, hence the matrix pair (A, B) is usually referred to as the *controllability pair* [13, 25]. The system (21) has the corresponding controllability pencil $\mathbf{S}_{\mathbf{C}}(s) = [sI_n + A \quad B]$. For the definition of controllability see any standard textbook on control theory, e.g., [26, 36].

7 Integer partitions and coins

In the next section we use integer partitions to represent the structural elements of a matrix or matrix pencil and coin moves to define the stratification rules. Here, we recall the definitions by quoting [10, 13].

An integer partition $\kappa = (\kappa_1, \kappa_2, ...)$ of an integer K is a monotonically non-increasing sequence of integers $(\kappa_1 \ge \kappa_2 \ge \cdots \ge 0)$ where $\kappa_1 + \kappa_2 + \cdots = K$. The union $\tau = (\tau_1, \tau_2, ...)$ of two integer partitions κ and ν is defined as $\tau = \kappa \cup \nu$ where $\tau_1 \ge \tau_2 \ge \cdots$. Furthermore, the conjugate partition of κ is defined as $\nu = \operatorname{conj}(\kappa)$, where ν_i is equal to the number of integers in κ that are equal to or greater than i, for i = 1, 2, ...

If ν is an integer partition, not necessarily of the same integer K as κ , and $\kappa_1 + \cdots + \kappa_i \geq \nu_1 + \cdots + \nu_i$ for $i = 1, 2, \ldots$, then $\kappa \geq \nu$. When $\kappa \geq \nu$ and $\kappa \neq \nu$ then $\kappa > \nu$. If κ, ν and τ are integer partitions of the same integer K and there does not exist any τ such that $\kappa > \tau > \nu$ where $\kappa > \nu$, then κ covers ν .

An integer partition $\kappa = (\kappa_1, \ldots, \kappa_n)$ can also be represented by *n* piles of coins, where the first pile has κ_1 coins, the second κ_2 coins and so on. An integer partition κ covers ν if ν can be obtained from κ by moving one coin one column rightward or one row downward, and keep κ monotonically non-increasing. Or equivalently, an integer partition κ is covered by τ if τ can be obtained from κ by moving one coin one column leftward or one row upward, and keep κ monotonically non-increasing. These two types of coin moves are defined in [10] and called minimum rightward and minimum leftward coin moves, respectively (see Figure 1).



Figure 1: Minimum leftward and rightward coin moves illustrate that $\kappa = (2, 2, 1, 1)$ is covered by $\tau = (3, 1, 1, 1)$ and $\kappa = (2, 2, 1, 1)$ covers $\nu = (2, 1, 1, 1, 1)$.

8 Structure integer partitions

We can now represent the structural elements of matrix pencils defined in Section 2 as integer partitions (notation from [13]):

(i) The column minimal indices as $\epsilon = (\epsilon_1, \ldots, \epsilon_{r_0})$, where $\epsilon_1 \ge \epsilon_2 \ge \cdots \ge \epsilon_{r_1} > \epsilon_{r_1+1} = \cdots = \epsilon_{r_0} = 0$. From the conjugate partition $(r_1, \ldots, r_{\epsilon_1}, 0, \ldots)$ of ϵ we define the integer partition $\mathcal{R}(sH + G) = (r_0) \cup (r_1, \ldots, r_{\epsilon_1})$.

(ii) The row minimal indices as $\eta = (\eta_1, \ldots, \eta_{l_0})$, where $\eta_1 \ge \eta_2 \ge \cdots \ge \eta_{l_1} > \eta_{l_1+1} = \cdots = \eta_{l_0} = 0$. From the conjugate partition $(l_1, \ldots, l_{\eta_1}, 0, \ldots)$ of η we define the integer partition $\mathcal{L}(sH + G) = (l_0) \cup (l_1, \ldots, l_{\eta_1})$.

(iii) For each distinct finite eigenvalue λ_i , i = 1, ..., q, with the finite elementary divisors on the form $(s - \lambda_i)^{h_1^{(i)}}, \ldots, (s - \lambda_i)^{h_{g_i}^{(i)}}$, where $h_1^{(i)} \ge \cdots \ge h_{g_i}^{(i)} \ge 1$, we introduce the integer partition $h_{\lambda_i} = (h_1^{(i)}, \ldots, h_{g_i}^{(i)})$ which is known as the Segre characteristic. The conjugate partition $\mathcal{J}_{\lambda_i}(sH + G) = (j_1, j_2, \ldots)$ of h_{λ_i} is the Weyr characteristic of λ_i .

(iv) For the infinite eigenvalue with the *infinite elementary divisors* on the form $\mu^{h_1}, \mu^{h_2}, \ldots, \mu^{h_{g_{\infty}}}$, with $h_1 \geq \cdots \geq h_{g_{\infty}} \geq 1$, we introduce the integer partition $h_{\infty} = (h_1, \ldots, h_{g_{\infty}})$ which is known as the *Segre characteristic* for the infinite eigenvalue. The conjugate partition $\mathcal{N}(sH+G) = (n_1, n_2, \ldots)$ of h_{∞} is the *Weyr characteristic* of the infinite eigenvalue.

The integer partitions above are referred to as *structure integer partitions* [13]. In addition, the condensed notation \mathcal{R} , \mathcal{L} , \mathcal{J} , and \mathcal{N} is used for the integer partitions corresponding to the right and left singular structures, and the Jordan structures of the finite and infinite eigenvalues, respectively, when it is obvious from the context.

9 The polynomial matrix space

Consider the $2d^2mn$ -dimensional space of $dm \times dn$ complex matrix pencils sH + G with Frobenius inner product

 $\langle sH + G, s\tilde{H} + \tilde{G} \rangle \equiv \operatorname{tr}(G\tilde{G}^* + H\tilde{H}^*)$. Let us for now only consider matrix pencils with d = 1 (see also [9, 10]). The *orbit* of an $m \times n$ matrix pencil:

$$\mathcal{O}(sH+G) = \{ U(sH+G)V^{-1} : \det(U) \cdot \det(V) \neq 0 \},$$
(22)

is the manifold of all equivalent matrix pencils, i.e., a manifold in the 2mn-dimensional space. All matrix pencils in the same orbit have the same canonical form, with the eigenstructure fixed. A *bundle* defines the union of all orbits with the same canonical form but with the eigenvalues unspecified, $\mathcal{B}(sH+G) := \bigcup_{\lambda_i} \mathcal{O}(sH+G)$ [1].

The dimension of $\mathcal{O}(sH+G)$ is equal to the dimension of the *tangent space* to $\mathcal{O}(sH+G)$ at sH+G:

$$\tan(sH + G) = \{sT_H + T_G = s(XH - HY) + (XG - GY)\},\$$

where $X \in \mathbb{C}^{m \times m}$ and $Y \in \mathbb{C}^{n \times n}$. The orthogonal complement of the tangent space is the normal space, nor $(sH + G) = \{sZ_H + Z_G\}$ where $Z_H H^* + Z_G G^* = 0$ and $H^*Z_H + G^*Z_G = 0$. The dimension of the normal space is called the *codimension* of $\mathcal{O}(sH + G)$, denoted by $\operatorname{cod}(sH + G)$. The codimension of the corresponding bundle is one less for each unspecified distinct eigenvalue. For example, a matrix pencil with k unspecified eigenvalues and the rest with known specified eigenvalues has $\operatorname{cod}(\mathcal{B}(sH + G)) = \operatorname{cod}(\mathcal{O}(sH + G)) - k$.

While a general matrix pencil of size $dm \times dn$ belongs to the complete pencil space, a polynomial matrix of degree d > 1 only resides in a subspace of the pencil space. An intuitive way to realize this is to consider, e.g., the right linearization $sH_r + G_r$ in (7) of a polynomial matrix. The right linearization is a matrix pencil with several fixed elements, where each fixed element decreases the degree of freedom by one. Following [11, 28], the set of $dm \times (dm - m + n) sH_r + G_r$ form a (d+1)mn-dimensional affine space in the pencil space, which is called the *Sylvester space* and denoted by $syl(sH_r + G_r)$. The manifold of equivalent polynomial matrices belong to the Sylvester space and is

$$\mathcal{O}(sH_r + G_r) = \{s\widetilde{H}_r + \widetilde{G}_r = U(sH_r + G_r)V^{-1} : s\widetilde{H}_r + \widetilde{G}_r \in \operatorname{syl}(sH_r + G_r) : \det(U) \cdot \det(V) \neq 0\}.$$

Lemma 5.3 shows that any full normal-rank polynomial matrix can be transformed such that P_d obtains full row-rank via a change of variables. Therefore, it follows that the codimension of $\mathcal{O}(sH_r + G_r)$

coincides with the codimension of the orbit of a matrix pair (A, B) associated with the system pencil $\mathbf{S}_{\mathrm{C}}(s)$ in (17) [13]:

$$\operatorname{cod}(sH_r + G_r) = c_{\operatorname{Right}} + c_{\operatorname{Jor}} + c_{\operatorname{Jor},\operatorname{Right}},\tag{23}$$

where

$$c_{\text{Right}} = \sum_{\epsilon_k > \epsilon_l} (\epsilon_k - \epsilon_l - 1), \ c_{\text{Jor}} = \sum_{i=1}^q \sum_{k=1}^{g_i} (2k - 1) h_k^{(i)}, \text{ and } c_{\text{Jor,Right}} = r_0 \sum_{i=1}^q \sum_{k=1}^{g_i} h_k^{(i)}.$$

The codimension of $\mathcal{O}(sH_{\ell} + G_{\ell})$ coincides with the codimension of the orbit of a matrix pair (A, C) associated with the system pencil $\mathbf{S}_{O}(s)$ in (17) [13]:

$$\operatorname{cod}(sH_{\ell} + G_{\ell}) = c_{\operatorname{Left}} + c_{\operatorname{Jor}} + c_{\operatorname{Jor},\operatorname{Left}}, \tag{24}$$

where

$$c_{\text{Left}} = \sum_{\eta_k > \eta_l} (\eta_k - \eta_l - 1), \ c_{\text{Jor}} = \sum_{i=1}^q \sum_{k=1}^{g_i} (2k-1)h_k^{(i)}, \text{ and } c_{\text{Jor,Left}} = l_0 \sum_{i=1}^q \sum_{k=1}^{g_i} h_k^{(i)}.$$

10 Stratifications

In this section, we present the stratification of orbits and bundles of full normal-rank polynomial matrices P(s).

The closure hierarchy of orbits (or bundles) is a stratification that we represent by a connected graph [10, 13]. The nodes of the graph correspond to orbits (or bundles) of canonical structures and the edges to their covering relations. The organization of the graph is from bottom to top (or top to bottom) with nodes in decreasing (or increasing) order of codimension. For an example see Figure 3.

Besides the orbit (or bundle) itself, the *closure* includes all orbits (or bundles) represented by the nodes which can be reached by a *downward path* in the graph. With a downward path we mean a path for which all edges start in a node and end in another node below in the graph. Similarly, a path in the opposite direction is called an *upward path*.

We remark that by adding a small perturbation to a matrix pencil (e.g., corresponding to a linearization of P(s)), it is always possible to make it more generic corresponding to a node along an upward path from the orbit (or bundle). In general, it is not possible to insist on a downward move by just adding a small perturbation of a given matrix pencil. However, the cases when a structure below in the hierarchy actually is nearby is often of particular interest, as it shows that a more degenerate structure can be found by a small perturbation. In a practical application this could mean that a controllable system is close to an uncontrollable one, which eventually could lead to a disaster.

By picking random matrix pencils of the same size, they will almost all have the same canonical structure, corresponding to the *most generic* case with the lowest codimension in the closure hierarchy. On the other side, the *most degenerate* case, or equivalently, the least generic case has the highest codimension. These extreme cases are represented by the topmost node (most generic) and the bottom node (least generic) in the closure hierarchy graph. For example, general rectangular matrix pencils may have several generic cases, but only one least generic case corresponding to a matrix pencil with only zero entries.

10.1 Known results for matrix pencils and matrix pairs

We recall the existing results for matrix pencils and matrix pairs in order to extend them to polynomial matrices in Section 10.2.

Theorem 10.1 below states the necessary and sufficient conditions for an orbit of two matrix pencils sH + G and $s\widetilde{H} + \widetilde{G}$ to be closest neighbours in a closure hierarchy, i.e. $\mathcal{O}(sH + G)$ covers $\mathcal{O}(s\widetilde{H} + \widetilde{G})$, where the orbit is the manifold of strictly equivalent matrix pencils (22). Notably, for the structure integer partition \mathcal{J}_{λ_i} the eigenvalue λ_i belongs to the extended complex plane $\overline{\mathbb{C}}$, i.e., $\lambda_i \in \mathbb{C} \cup \{\infty\}$. Furthermore, the restrictions on r_0 and l_0 in rules 1 and 2 correspond to that the number of L_k and L_k^T blocks cannot change. The corresponding set of rules for bundles of matrix pencils is derived in [10].

Theorem 10.1 [10] Given the structure integer partitions \mathcal{L} , \mathcal{R} , and \mathcal{J}_{λ_i} of sH + G, where $\lambda_i \in \overline{\mathbb{C}}$, one of the following if-and-only-if rules finds $s\widetilde{H} + \widetilde{G}$ fulfilling orbit covering relations with $s\widetilde{H} + \widetilde{G}$:

- (1) Minimum rightward coin move in \mathcal{R} (or \mathcal{L}).
- (2) If the rightmost column in \mathcal{R} (or \mathcal{L}) is one single coin, move that coin to a new rightmost column of some \mathcal{J}_{λ_i} (which may be empty initially).
- (3) Minimum leftward coin move in any \mathcal{J}_{λ_i} .
- (4) Let k denote the total number of coins in all of the longest (= lowest) rows from all of the \mathcal{J}_{λ_i} . Remove these k coins, add one more coin to the set, and distribute k + 1 coins to r_p , $p = 0, \ldots, t$ and l_q , $q = 0, \ldots, k - t - 1$ such that at least all nonzero columns of \mathcal{R} and \mathcal{L} are given coins.

Rules 1 and 2 are not allowed to make coin moves that affect r_0 (or l_0).

The next theorem states the necessary and sufficient conditions for an orbit of two controllability pairs to be closest neighbours in a closure hierarchy, where we consider the orbit under feedback equivalence:

$$\mathcal{O}(A,B) = \left\{ P \begin{bmatrix} sI + A & B \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ R & Q^{-1} \end{bmatrix} : \det(P) \cdot \det(Q) \neq 0 \right\}$$

Theorem 10.2 [13] Given the structure integer partitions \mathcal{R} and \mathcal{J}_{λ_i} of a controllability pair (A, B), one of the following if-and-only-if rules finds $(\widetilde{A}, \widetilde{B})$ fulfilling orbit covering relations with (A, B).

- (1) Minimum rightward coin move in \mathcal{R} .
- (2) If the rightmost column in \mathcal{R} is one single coin, move that coin to a new rightmost column of some \mathcal{J}_{λ_i} (which may be empty initially).
- (3) Minimum leftward coin move in any \mathcal{J}_{λ_i} .

Rules 1 and 2 are not allowed to do coin moves that affect r_0 .

The corresponding set of rules for bundles of controllability pairs and orbits and bundles for observability pairs are also derived in [13].

Theorem 10.3 [18, 40] Let $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times m}$, then the most generic structure of the controllability pair (A, B) has $\mathcal{R} = (r_0, \ldots, r_\alpha, r_{\alpha+1})$ where $r_0 = \cdots = r_\alpha = m$, $r_{\alpha+1} = n \mod m$, and $\alpha = \lfloor n/m \rfloor$. The least generic controllability pair has $m \ L_0$ blocks and n Jordan blocks of size 1×1 corresponding to an eigenvalue of multiplicity n.

10.2 Stratification of polynomial matrices

As we have shown in Section 6, right and left linearizations of full normal-rank polynomial matrices can be expressed, respectively, as the system pencils $\mathbf{S}_{\mathrm{R}}(s)$ and $\mathbf{S}_{\mathrm{L}}(s)$ in (16). However, these linearizations result in a system pencil with a very special structure. We will now show that the covering relations between orbits and bundles of polynomial matrices can be derived from the covering relations for general matrix pencils. To do this we first state the following lemma.

Lemma 10.4 Let $sH_r + G_r$ be the right linearization (7) of an $m \times n$ polynomial matrix P(s) of normalrank m. A perturbation in the non-fixed elements of $sH_r + G_r$ corresponds to a perturbation in the coefficient matrices of P(s). Moreover, the corresponding perturbed $sH_r + G_r$ and $\tilde{P}(s)$ have the same structural elements.

A similar relation holds between a polynomial matrix of normal-rank n and its left linearization $sH_{\ell} + G_{\ell}$.

Proof. We prove the lemma for $m \times n$ polynomial matrices P(s) of normal-rank m.

Assume that $sH_r + G_r$ is the right linearization of P(s), i.e., they have the same structural elements. Let $s\tilde{H}_r + \tilde{G}_r$ be a perturbed pencil of $sH_r + G_r$ for which the structural elements differ from them of $sH_r + G_r$. Note, only the non-fixed elements in $sH_r + G_r$ are allowed to be perturbed. From Theorem 3.5 we have that there exits a polynomial matrix $\tilde{P}(s)$ with the same structural elements as $s\tilde{H}_r + \tilde{G}_r$ which is also the right linearization of $\tilde{P}(s)$. Therefore, a structured perturbation in the linearization of P(s) corresponds to a perturbation in P(s). \Box

polynomial matrix $P(s)$, one of the following <i>if-and-only-if</i> rules finds $sH_r + G_r$ fulfilling orbit or bundle	
covering relations with $sH_r + G_r$.	
A. $\mathcal{O}(sH_r + G_r)$ covers $\mathcal{O}(s\tilde{H}_r + \tilde{G}_r)$:	B. $\mathcal{O}(sH_r + G_r)$ is covered by $\mathcal{O}(s\widetilde{H}_r + \widetilde{G}_r)$:
 Minimum rightward coin move in <i>R</i>. If the rightmost column in <i>R</i> is one single coin, move that coin to a new rightmost column of some <i>J</i>_{λi} (which may be empty initially). Minimum leftward coin move in any <i>J</i>_{λi} as long as <i>j</i>₁⁽ⁱ⁾ does not exceed <i>m</i>. Rules 1 and 2 are not allowed to do coin moves that affect <i>r</i>₀. 	 Minimum leftward coin move in <i>R</i>, without affecting r₀. If the rightmost column in some <i>J</i>_{λi} consists of one coin only, move that coin to a new rightmost column in <i>R</i>. Minimum rightward coin move in any <i>J</i>_{λi}.
C. $\mathcal{B}(sH_r + G_r)$ covers $\mathcal{B}(s\widetilde{H}_r + \widetilde{G}_r)$:	D. $\mathcal{B}(sH_r + G_r)$ is covered by $\mathcal{B}(s\widetilde{H}_r + \widetilde{G}_r)$:
(1) Same as rule 1 above.	(1) Same as rule 1 above.
(2) Same as rule 2 above, except it is only allowed to start a new set corresponding to a new eigenvalue (i.e., no appending to non-empty sets).	 (2) Same as rule 2 above, except that J_{λi} must consist of one coin only. (3) Same as rule 3 above
(3) Same as rule 3 above.	(4) For any \mathcal{J}_{λ_i} , divide the set of coins into two new
(4) Let any pair of eigenvalues coalesce, i.e., take the union of their sets of coins.	partitions so that their union is \mathcal{J}_{λ_i} .

Table 1: Given the structure integer partitions \mathcal{R} and \mathcal{J}_{λ_i} of $sH_r + G_r$ associated with a full normal-rank

Remark 10.1 Lemma 10.4 implicitly shows that the map between the orbit spaces of $sH_r + G_r$ and P(s) is a homeomorphism, i.e., the continuous map $f : \mathcal{O}(P(s)) \to \mathcal{O}(sH_r + G_r)$ is a bijection. See also [32].

Lemma 10.4 allows us to formulate the covering relations of an $m \times n$ full normal-rank polynomial matrix P(s) in terms of coin rules on the structure integer partitions of its linearization. For consistency with earlier published results in [10, 13] our new findings in Theorem 10.5 (with Table 1) and colloraries are stated using the same notation and similar formulations. Notably, the eigenvalue λ_i corresponding to the structure integer partition \mathcal{J}_{λ_i} belongs to $\overline{\mathbb{C}}$.

Theorem 10.5 Let P(s) be an $m \times n$ polynomial matrix of exact degree d and normal-rank m, and let $sH_r + G_r$ be its right linearization (7). Given the structure integer partitions \mathcal{R} and \mathcal{J}_{λ_i} of $sH_r + G_r$, where $\lambda_i \in \overline{\mathbb{C}}$, one of the if-and-only-if rules of A-D in Table 1 finds $s\widetilde{H}_r + \widetilde{G}_r$ fulfilling orbit or bundle covering relations with $sH_r + G_r$.

The rules for the dual left linearization $sH_{\ell} + G_{\ell}$ (11), associated with the polynomial matrix P(s) of normal-rank n, are obtained by exchanging \mathcal{R} with \mathcal{L} and m with n in Table 1.

Proof. The new restrictions in the rules, with respect to the cover rules in Theorem 10.1 for general matrix pencils sH + G, follow directly from Theorem 5.1. The restrictions are: No L^T blocks can exist (rule (4) in Theorem 10.1 cannot be applied). Since there can at most be m finite and m infinite elementary divisors, $j_1^{(i)}$ in \mathcal{J}_{λ_i} for any $\lambda_i \in \overline{\mathbb{C}}$ can at most be m. Finally, r_0 in \mathcal{R} must be n - m, which implies that the number of L blocks remains fixed and is n - m. \Box

Comments to Table 1: The restriction for rules $A_{\cdot}(1)$ and $A_{\cdot}(2)$ implies that the number of right singular blocks remain fixed, and rule $A_{\cdot}(3)$ corresponds to the nilpotent case.

It is now straightforward to derive the cover relations for square full normal-rank polynomial matrices. Notably, these rules coincide with the cover rules for matrices [1, 10] with the exception of that the number of Jordan blocks is restricted by the normal-rank.

Corollary 10.6 Let P(s) be an $m \times m$ polynomial matrix of exact degree d and normal-rank m, and let $sH_r + G_r$ be its right linearization (7). Given the structure integer partition \mathcal{J}_{λ_i} of $sH_r + G_r$, where $\lambda_i \in \mathbb{C}$, rule (3) of A-B and rules (3) and (4) of C-D in Table 1 find $sH_r + \tilde{G}_r$ fulfilling orbit or bundle covering relations with $sH_r + G_r$.

The rules for the dual left linearization $sH_{\ell} + G_{\ell}$ (11), associated with the polynomial matrix P(s) of normal-rank n, are obtained by the rules specified above where m is exchanged with n.

Proof. The proof follows directly from Theorem 10.5 together with the restrictions of Corollary 5.2. \Box

The following theorem gives the cover relations for full normal-rank polynomial matrices with a full rank highest degree coefficient matrix. Note that the eigenvalues λ_i corresponding to the structure integer partitions \mathcal{J}_{λ_i} are all finite.

Corollary 10.7 Let $P(s) = P_d s^d + \ldots + P_1 s + P_0$ be an $m \times n$ polynomial matrix of exact degree d, normal-rank m, and with P_d of full row rank. Furthermore, let $sH_r + G_r$ be its right linearization (7), where $H_r \equiv \begin{bmatrix} I_{dm} & 0_{dm \times n-m} \end{bmatrix}$. Given the structure integer partitions \mathcal{R} and \mathcal{J}_{λ_i} of $sH_r + G_r$, where $\lambda_i \in \mathbb{C}$, one of the if-and-only-if rules of A-D in Table 1 finds $s\tilde{H}_r + \tilde{G}_r$ fulfilling orbit or bundle covering relations with $sH_r + G_r$.

The rules for the dual left linearization $sH_{\ell} + G_{\ell}$ (11), associated with the polynomial matrix P(s) of normal-rank n and with P_d of full column rank, are obtained by exchanging \mathcal{R} with \mathcal{L} and m with n in Table 1.

Proof. The proof can be derived from Corollary 5.4 together with either Theorem 10.2 for matrix pairs or Theorem 10.5 for matrix pencils.

The only restriction, with respect to the cover rules in Theorem 10.2, is that $j_1^{(i)}$ for any eigenvalue λ_i can at most be m (or n if normal-rank is n).

The only restriction, with respect to the cover rules in Theorem 10.5, is that there cannot exist infinite eigenvalues since H_r has full row rank (or H_ℓ has full column rank if normal-rank is n). \Box

The canonical structure elements of the most and least generic orbits or bundles in the stratification of a full normal-rank polynomial matrix are given by the next theorem. Notably, the most generic $m \times n$ cases correspond to most generic matrix pairs, while the least generic (most degenerate) cases do not have a similar correspondence.

Theorem 10.8 Let P(s) be an $m \times n$ polynomial matrix of exact degree d and full normal-rank r, where $m \neq n$.

If r = m, the most generic orbit (or bundle) of P(s) has the structure integer partition $\mathcal{R} = (r_0, \ldots, r_\alpha, r_{\alpha+1})$ where $r_0 = \cdots = r_\alpha = n - m$, $r_{\alpha+1} = (dm) \mod (n-m)$, and $\alpha = \lfloor (dm)/(n-m) \rfloor$.

There exist several least generic orbits with the same codimension. The least generic orbits have $\mathcal{R} = (n-m)$ and \mathcal{J}_{λ_i} , i = 1, ..., d, which are constructed as follows. Let $\kappa = (k_1, ..., k_d) = (m, ..., m)$. Divide the integer partition κ between \mathcal{J}_{λ_i} , i = 1, ..., d, such that their union is κ . One or several \mathcal{J}_{λ_i} can be empty.

The least generic bundle has $\mathcal{R} = (n - m)$ and $\mathcal{J}_{\lambda} = (j_1, \ldots, j_d) = (m, \ldots, m)$, i.e., $m J_d$ blocks corresponding to a single eigenvalue of multiplicity m.

If r = n, the most and least generic orbits (or bundles) of P(s) are obtained by exchanging \mathcal{R} with \mathcal{L} and interchanging m and n in the above expressions.

Proof. Most generic: Since the codimension (23) coincides with the codimension for matrix pairs (A, B) if normal-rank is m (or (A, C) if normal-rank is n), it follows that the most generic orbit and bundle have the same canonical forms as the corresponding matrix pairs [40, 13] (no restrictions on the structural elements exist).

Least generic orbit: Theorem 5.1 states that all structural elements add up to dm. Since all right minimal indices $\epsilon_i = 0$ (i.e., only L_0 blocks exist) they make no contribution to the sum (12). Consequently, $\sum h_j^{(i)} = dm$, where one of the eigenvalues may be infinite. Moreover, dm is an integer and we have at most m nonzero elementary divisors for each eigenvalue, therefore there are at most d eigenvalues. Two extreme cases exist: (i) only one eigenvalue λ_1 , which means that the canonical form must be m Jordan blocks of size $d \times d$ ($\mathcal{J}_{\lambda_1} = (m, \ldots, m)$); (ii) d distinct eigenvalues λ_i , which means that each of them has m Jordan blocks of size 1×1 ($\mathcal{J}_{\lambda_i} = (m)$, $i = 1, \ldots, d$). Since the sum of all elementary divisors $\sum h_j^{(i)} = dm$, the number of (distinct) eigenvalues does not change the codimension. Therefore, all possible splittings of $\kappa = (k_1, \ldots, k_d) = (m, \ldots, m)$ into one or several \mathcal{J}_{λ_i} , $i = 1, \ldots, d$, are possible. Indeed, the number of least generic orbits is equal to the number of combinations in the dominance ordering of the integer d.

In the bundle case, the least generic bundle has only one multiple eigenvalue corresponding to $\mathcal{J}_{\lambda} = \kappa$.

Finally, we consider square $m \times m$ polynomial matrices P(s) of exact degree d and with full normalrank r = m, and only illustrate here with and example where m = d = r = 2. There are five most generic cases: $\mathcal{O}(J_4(\mu_1)), \mathcal{O}(J_3(\mu_1) \oplus J_1(\mu_2)), \mathcal{O}(J_2(\mu_1) \oplus J_2(\mu_2)), \mathcal{O}(J_2(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_3))$, and $\mathcal{O}(J_1(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_3) \oplus J_1(\mu_4))$. Each of them are independent, i.e. they belong to different closure hierarchy graphs with the following least generic orbits: $\mathcal{O}(2J_2(\mu_1)), \mathcal{O}(J_2(\mu_1) \oplus J_1(\mu_1) \oplus J_1(\mu_2))$, $\mathcal{O}(2J_1(\mu_1)\oplus 2J_1(\mu_2)), \mathcal{O}(2J_1(\mu_1)\oplus J_1(\mu_2)\oplus J_1(\mu_3)), \text{ and } \mathcal{O}(J_1(\mu_1)\oplus J_1(\mu_2)\oplus J_1(\mu_3)\oplus J_1(\mu_4)), \text{ respectively.}$ We remark that the last of the five cases above, indeed, is a stratification with only one node. The stratification of the five cases can be obtained from the closure hierarchy graph of a regular matrix pencil, with the restriction that it at most can be *m* elementary divisors associated with each eigenvalue.

11 Examples

To illustrate the stratification theory we make use of two mechanical systems. The first is a uniformed platform and the second is a half-car suspension model. The software tool StratiGraph [24, 23] is used to generate and visualize the closure hierarchy graphs.

Before we move on to the examples we introduce a condensed notation for the KCF, used in Strati-Graph. A general block diagonal matrix $A = \text{diag}(A_1, A_2, \ldots, A_b)$ with b blocks can be represented as a direct sum

$$A \equiv A_1 \oplus A_2 \oplus \dots \oplus A_b \equiv \bigoplus_{k=1}^b A_k.$$

Now, the KCF (6) can compactly be expressed as

$$U(sH+G)V^{-1} \equiv \mathbb{L} \oplus \mathbb{L}^T \oplus \mathbb{J}(\lambda_1) \oplus \cdots \oplus \mathbb{J}(\lambda_q) \oplus \mathbb{N}_q$$

where

$$\mathbb{L} = \bigoplus_{k=1}^{r_0} L_{\epsilon_k}, \quad \mathbb{L}^T = \bigoplus_{k=1}^{l_0} L_{\eta_k}^T, \quad \mathbb{J}(\lambda_i) = \bigoplus_{k=1}^{g_i} J_{h_k}(\lambda_i), \quad \text{and} \quad \mathbb{N} = \bigoplus_{k=1}^{g_\infty} N_{h_k}.$$

Notice that blocks of the KCF in the direct sum notation above are, without loss of generality, ordered so that the singular blocks (\mathbb{L} and \mathbb{L}^T) appear first.

11.1 Uniform platform

The first example consists of a thin uniform platform supported at both ends by springs [13, 31]. The platform has mass m and length 2l, and the springs have elasticity coefficients k_1 , k_2 and viscous damping coefficients c_1 , c_2 , see Figure 2. The vertical position z of the platform's center, the angle φ , and their



Figure 2: Mechanical system consisting of a uniform platform controlled by a vertical force [13, 31].

velocities can be controlled by the vertical force F. The force is applied vertically onto the platform at a distance Δl , $-1 \leq \Delta \leq 1$, from the center of the platform.

In [13], the controllability pair corresponding to the linearized model of the uniform platform was analyzed using stratifications. Here we will study the system using its polynomial matrix representation. The equations of motion linearized near the equilibrium are

$$m\ddot{z} + (c_1 + c_2)\dot{z} + (k_1 + k_2)z + l(c_1 - c_2)\dot{\varphi} + l(k_1 - k_2)\varphi = F,$$

$$J\ddot{\varphi} + l(c_1 - c_2)\dot{z} + l(k_1 - k_2)z + l^2(c_1 + c_2)\dot{\varphi} + l^2(k_1 + k_2)\varphi = -\Delta lF,$$
(25)



Figure 3: The graph shows the complete bundle stratification of a 4×5 system pencil associated with a general matrix pair (A, B). The light grey area marks the complete bundle stratification of the right linearization (27), and the dark grey area the only possible bundles for the uniform platform. The numbers on the left are the codimensions of the bundles on each level, which are the same for the system bundles and the corresponding linearizations of the polynomial matrix representations.

where $J = ml^2/3$ is the moment of inertia of the platform around the gravity center. This system can be written as a second-order differential equation on the form

$$M\ddot{x} + C\dot{x} + Kx = Eu, \text{ where} M = \begin{bmatrix} m & 0\\ 0 & J \end{bmatrix}, C = \begin{bmatrix} c_1 + c_2 & l(c_1 - c_2)\\ l(c_1 - c_2) & l^2(c_1 + c_2) \end{bmatrix}, K = \begin{bmatrix} k_1 + k_2 & l(k_1 - k_2)\\ l(k_1 - k_2) & l^2(k_1 + k_2) \end{bmatrix}, E = \begin{bmatrix} 1\\ -\Delta l \end{bmatrix}, x = \begin{bmatrix} z\\ \varphi \end{bmatrix}, \text{ and } u = F.$$
(26)

By definition, the mass matrix M is non-singular and $M^{-1} = \text{diag}(m^{-1}, J^{-1})$, hence the associated polynomial matrix has full normal-rank r = 2 and the highest degree coefficient matrix is non-singular. Using the technique in Example 6.2, the right linearization of the associated 2×3 polynomial matrix of (26) is

ſ

$$sI_4 + A \mid B] = \begin{bmatrix} sI_2 & M^{-1}K \\ -I_2 & sI_2 + M^{-1}C \end{bmatrix} \begin{pmatrix} M^{-1}E \\ 0 \end{bmatrix}.$$
 (27)

The complete bundle stratification of a general system pencil $\begin{bmatrix} sI_4 + A & B \end{bmatrix}$ of size 4×5 is shown in



Figure 4: Half-car passive suspension model.

Figure 3. However, since the dynamics of the platform can be represented by a polynomial matrix we can instead use Corollary 10.7. This reveals that the stratification of (27) only consists of the bundles highlighted by the light grey area in Figure 3. Notably, the bundle for the system pencil and the corresponding bundle (with the same canonical form) for the linearization have in general different dimensions, but their codimensions are the same. Additionally, not all bundles or part of these exist for this specific example due to the fixed 1 (one) in the *E* matrix of (26). The least generic possible structure can be determined by setting the parameters c_1, c_2, k_1, k_2 , and Δ to zero (*l* and *m* must remain non-zero), which restricts the number of possible bundles to the four highlighted by the dark grey area (see also the discussion in [13]). The most generic $\mathcal{B}(L_4)$ corresponds to a controllable system, while the bundles with codimension 1 and 2 correspond to uncontrollable systems with one and two uncontrollable modes. Finally, the system associated with $\mathcal{B}(L_2 \oplus J_2(\mu_1))$ of codimension 3 has two multiple uncontrollable modes.

11.2 Half-car suspension model

n 1

The second example describes the half-car passive suspension model with four degrees of freedom shown in Figure 4, where k_i are stiffnesses, c_i dampings, l_i lengths, m_i masses, and $J_p \approx m_b l_f l_r$ is the body moment of inertia. The model represents one side of a car (front and rear suspension), where the pitch φ and heave motion z_b of the vehicle body and the vertical translation of the front and rear axles (z_f and z_r , respectively) can be analyzed. Typical values for a passenger sedan can be found in, e.g., [35].

The equations of motion of the half-car suspension model are:

$$m_f \ddot{z}_f = k_{t_f} (z_f - q_f) + k_f (z_b - \varphi l_f - z_f) + c_f (\dot{z}_b - \dot{\varphi} l_f - \dot{z}_f) + m_f g,$$
(28)

$$n_r \ddot{z}_r = k_{t_r} (z_r - q_r) + k_r (z_b + \varphi l_r - z_r) + c_r (\dot{z}_b + \dot{\varphi} l_r - \dot{z}_r) + m_r g,$$
(29)

$$n_b \ddot{z}_b = k_f (z_f - z_b + \varphi l_f) + k_r (z_r - z_b + \varphi l_r) + c_f (\dot{z}_f - \dot{z}_b + \dot{\varphi} l_f)$$

$$+ c_r (\dot{z}_r - \dot{z}_b - \dot{\varphi} l_r) + m_b q,$$
(30)

$$J_p \ddot{\varphi} = -k_f l_f (z_f - z_b + \varphi l_f) - c_f l_f (\dot{z}_f - \dot{z}_b + \dot{\varphi} l_f) + k_r l_r (z_r - z_b + \varphi l_r)$$

$$+ c_r l_r (\dot{z}_r - \dot{z}_b - \dot{\varphi} l_r).$$

$$(31)$$

Let the state vector be $x = \begin{bmatrix} z_f & z_r & z_b \end{bmatrix}^T$, and the input vector be $u = \begin{bmatrix} q_f & q_r & g \end{bmatrix}^T$, where q_f , q_r are the road heights. Then the equations (28)–(30) can be represented in matrix form by the second-order differential equation

$$M\ddot{x} + C\dot{x} - Kx - C_p\dot{\varphi} - K_p\varphi = Eu, \qquad (32)$$

and (31) as

$$J_p \ddot{\varphi} + k_p \dot{\varphi} + c_p \varphi - C_p^T \dot{x} - K_p^T x = 0, \qquad (33)$$

where

$$M = \operatorname{diag}(m_{f}, m_{r}, m_{b}), \qquad E = \begin{bmatrix} 1 & 0 & m_{f} \\ 0 & 1 & m_{r} \\ 0 & 0 & m_{b} \end{bmatrix}, \\ C = \begin{bmatrix} c_{f} & 0 & -c_{f} \\ 0 & c_{r} & -c_{r} \\ -c_{f} & -c_{r} & c_{f} + c_{r} \end{bmatrix}, \qquad K = \begin{bmatrix} k_{t_{f}} - k_{f} & 0 & k_{f} \\ 0 & k_{t_{r}} - k_{r} & k_{r} \\ k_{f} & k_{r} & -k_{f} - k_{r} \end{bmatrix}, \\ C_{p} = \begin{bmatrix} -c_{f}l_{f} & c_{r}l_{r} & c_{f}l_{f} - c_{r}l_{r} \end{bmatrix}^{T}, \qquad K_{p} = \begin{bmatrix} -k_{f}l_{f} & k_{r}l_{r} & k_{f}l_{f} + k_{r}l_{r} \end{bmatrix}^{T}, \\ c_{p} = c_{f}l_{f}^{2} + c_{r}l_{r}^{2}, \qquad \text{and} \qquad k_{p} = k_{f}l_{f}^{2} + k_{r}l_{r}^{2}.$$

Using the Laplace variable s, (32) and (33) can be expressed as

$$Ms^{2}x + Csx - Kx - C_{p}s\varphi - K_{p}\varphi = Eu, \text{ and}$$

$$J_{p}s^{2}\varphi + k_{p}s\varphi + c_{p}\varphi - C_{p}^{T}sx - K_{p}^{T}x = 0,$$

(34)

respectively. Eliminating φ from (34) leads to the fourth-order differential equation

$$P_4 x^{(4)} + P_3 x^{(3)} + P_2 x^{(2)} + P_1 x^{(1)} + P_0 x = Q_2 u^{(2)} + Q_1 u^{(1)} + Q_0 u,$$
(35)

where

$$\begin{aligned} P_4 &= J_p M, & P_3 &= k_p M + J_p C, \\ P_2 &= k_p C + c_p M - J_p K - C_p C_p^T, & P_1 &= c_p C - k_p K - K_p C_p^T - C_p K_p^T, \\ P_0 &= c_p K - K_p K_p^T, & Q_2 &= J_p E, & Q_1 &= k_p E, \text{ and } & Q_0 &= c_p E. \end{aligned}$$

Using the technique outlined in Section 6 and assuming that J_p and M are non-singular, the right linearization of the associated 3×6 polynomial matrix of (35) is

$$\begin{bmatrix} sI_{12} + A \mid B \end{bmatrix} = \begin{bmatrix} sI_3 & P_4^{-1}P_0 \\ -I_3 & sI_3 & P_4^{-1}P_1 \\ & -I_3 & sI_3 & P_4^{-1}P_2 \\ & & -I_3 & sI_3 + P_4^{-1}P_3 \end{bmatrix} \begin{pmatrix} P_4^{-1}Q_0 \\ P_4^{-1}Q_1 \\ P_4^{-1}Q_2 \\ 0 \end{bmatrix},$$
(36)

where $A \in \mathbb{C}^{12 \times 12}$ and $B \in \mathbb{C}^{12 \times 3}$.

The complete stratification of (36) has 6416 different orbits! Instead of computing the complete graph we only derive the subgraph shown in Figure 5. The graph represents all the controllable orbits (on the left) together with the closest uncontrollable orbits with one uncontrollable mode (on the right) which can be reached by a perturbation of the polynomial matrix coefficients. The most generic orbit with KCF $3L_4$ corresponds to the case when the three transformed inputs $\{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3\}$ in the linearization control four states each, while the least generic controllable orbit ($\mathcal{O}(L_{12} \oplus 2L_0)$) with codimension 22) corresponds to when \tilde{u}_1 controls all twelve states. And finally, one example taken from the uncontrollable part of the graph is when the two suspensions do not have any damping ($c_f = c_r = 0$). Such a configuration belongs to $\mathcal{O}(L_5 \oplus 2L_3 \oplus J_1(\mu_1))$ with codimension 6. In practice, this means that the corresponding orbit for a suspension system with low damping factor is likely to be close to $\mathcal{O}(L_5 \oplus 2L_3 \oplus J_1(\mu_1))$.

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Figure 5: Subgraph of the complete orbit stratification of the polynomial system (36). The nodes in the right-most part of the graph represent the orbits of uncontrollable systems with one uncontrollable mode.

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